PRODUCTS ON SCHATTEN-VON NEUMANN CLASSES AND MODULATION SPACES

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ABSTRACT. We consider modulation space and spaces of Schatten-von Neumann symbols where corresponding pseudo-differential operators map one Hilbert space to another. We prove Hölder-Young and Young type results for such spaces under dilated convolutions and multiplications. We also prove continuity properties for such spaces under the twisted convolution, and the Weyl product. These results lead to continuity properties for twisted convolutions on Lebesgue spaces, e. g. $L^p_{(\omega)}$ is a twisted convolution algebra when $1 \le p \le 2$ and appropriate weight ω.

0. INTRODUCTION

In this paper we establish continuity properties for various products on modulation spaces and a familly of symbol classes such that corresponding pseudo-differential operators are of Schatten-von Neumann types.

This means that each symbol class consists of all tempered distributions such that the corresponding pseudo-differential operators are Schatten-von Neumann operators of certain degree from one Hilbert space to another. For such spaces of functions and distributions, we establish Young type and Hölder-Young type inequalities with respect to the Weyl product, twisted convolution, dilated convolutions and dilated multiplications. These products are important in the theory of pseudo-differential operators. In fact, the Weyl product corresponds to compositions of Weyl operators on the symbol side. On the symplectic Fourier transform side, the Weyl product takes the form as a twisted convolution. In the theory of pseudo-differential operators, it is in many situations convenient to approximate a pseudo-differential operator with a Toeplitz operator. Then the Weyl symbol of a Toeplitz operator is an ordinary convolution of the Toeplitz symbol and a rank one element, which can be rewritten as a product between symbols by using the symplectic Fourier transform.

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In particular we generalize results in [43,45], where similar questions were considered for classical modulation spaces, and spaces of pseudo-differential operators of Schatten-von Neumann types on L^2. The Weyl product in the context of modulation space theory was recently investigated in [26]. From these results we establish continuity for the twisted convolutions when acting on modulation spaces of Wiener amalgam type, using the fact that the Fourier transform of a Weyl product is essentially a twisted convolution of the Weyl symbols. From these results we thereafter prove continuity properties of the twisted convolution on weighted Fourier Lebesgue spaces.

We also apply our results on Toeplitz operators and prove that for each appropriate dilated symbol to a Schatten-von Neumann pseudo-differential operator, then the Toeplitz operator belongs the same Schatten-von Neumann class.

In order to be more specific, we recall some definitions. Assume that $t \in \mathbf{R}$ is fixed and that $a \in \mathscr{S}(\mathbf{R}^{2d})$. (We use the same notation for the usual function and distribution spaces as in [27].) Then the pseudo-differential operator $a_t(x, D)$ with symbol a is the continuous operator on $\mathscr{S}(\mathbf{R}^d)$, defined by the formula

$$
\begin{aligned}
(a_t(x,D)f)(x) &= (\operatorname{Op}_t(a)f)(x) \\
&= (2\pi)^{-d} \iint a((1-t)x+ty,\xi)f(y)e^{i\langle x-y,\xi\rangle}\,dyd\xi.
\end{aligned} \tag{0.1}
$$

The definition of $a_t(x, D)$ extends to each $a \in \mathscr{S}'(\mathbf{R}^{2d})$, and then $a_t(x, D)$ is continuous from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$. (Cf. e. g. [27].) In fact, the Fourier transform $\mathscr{F}$ is the linear and continuous operator on $\mathscr{S}'(\mathbf{R}^d)$ which takes the form

$$
(\mathscr{F}f)(\xi) = \widehat{f}(\xi) = (2\pi)^{-d/2} \int f(x)e^{i\langle x,\xi\rangle}\,dx, \tag{0.2}
$$

when $f \in L^1(\mathbf{R}^d)$. If $\mathscr{F}_2 F$ is the partial Fourier transform of $F(x, y) \in \mathscr{S}'(\mathbf{R}^{2d})$ with respect to the y variable and $a \in \mathscr{S}'(\mathbf{R}^{2d})$, then we let $a_t(x, D)$ be the linear and continuous operator from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ with the distribution kernel

$$
K_{a,t}(x,y) = (\mathscr{F}_2^{-1}a)((1-t)x+ty, x-y). \tag{0.3}
$$

If $t = 1/2$, then $a_t(x, D)$ is equal to the Weyl operator $a^w(x, D)$ for a. If instead $t = 0$, then the standard (Kohn-Nirenberg) representation $a(x, D)$ is obtained.

For each $a \in \mathscr{S}(\mathbf{R}^{2d})$ we also let Aa be the linear and continuous operator from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}(\mathbf{R}^d)$, given by

$$
(Aa)(x,y) = (2\pi)^{-d/2} \int a((y-x)/2,\xi)e^{-i\langle x+y,\xi\rangle}\,d\xi. \tag{0.4}
$$

(Here and in what follows we identify operators with their distribution kernels.) This operator representation is closely related to the Weyl

quantization in the sense that

$$Aa(x, y) = (2\pi)^{d/2} K_{a,1/2}(-x, y) = (2\pi)^{d/2} K_{\mathscr{F}_\sigma a,1/2}(x, y),$$

where $\mathscr{F}_\sigma$ denotes the symplectic Fourier transform (see Section 1 for strict definitions). Here the first equality follows from (0.3) and (0.4). In particular, the definition of Aa extends to each $a \in \mathscr{S}'(\mathbf{R}^{2d})$ and then Aa is continuous from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$, since the same is true for the Weyl quantization.

In the Weyl calculus, operator composition corresponds on the symbol level to the *Weyl product*, sometimes also called the twisted product, denoted by $\#$. In Sections 2–3 we establish extensions for the Weyl product and twisted convolutions on modulation spaces, which we shall describe now.

The modulation spaces was introduced by Feichtinger in [10], and developed further and generalized in [11, 14–16, 21], where Feichtinger and Gröchenig established the theory of coorbit spaces. In particular, the modulation spaces $M^{p,q}_{(\omega)}$ and $W^{p,q}_{(\omega)}$, where ω denotes a weight function on phase (or time-frequency shift) space, appear as the set of tempered (ultra-) distributions whose STFT belong to the weighted and mixed Lebesgue space $L^{p,q}_{1,(\omega)}$ and $L^{p,q}_{2,(\omega)}$ respectively. (See Section 1 for strict definitions.) It follows that ω, p and q to some extent quantify the degrees of asymptotic decay and singularity of the distributions in $M^{p,q}_{(\omega)}$ and $W^{p,q}_{(\omega)}$. By choosing the weight ω in appropriate ways, the space $W^{p,q}_{(\omega)}$ becomes a Wiener amalgam space, introduced by Feichtinger in [8]. Furthermore, if ω is trivial, i. e. $\omega = 1$, then $M^{p,q}_{(\omega)}$ is the *classical* modulation space $M^{p,q}$. (See [12] for the most updated description of modulation spaces.)

From the construction of these spaces, it turns out that modulation spaces of the form $M^{p,q}_{(\omega)}$ and Besov spaces in some sense are rather similar, and sharp embeddings between these spaces can be found in [45, 47], which are improvements of certain embeddings in [19]. (See also [38] for verification of the sharpness.) In the same way it follows that modulation spaces of the form $W^{p,q}_{(\omega)}$ and Triebel-Lizorkin spaces are rather similar.

During the last 15 years many results have been proved which confirm the usefulness of the modulation spaces and their Fourier transforms in time-frequency analysis, where they occur naturally. For example, in [16, 21], it is shown that all such spaces admit reconstructible sequence space representations using Gabor frames.

Parallel to this development, modulation spaces have been incorporated into the calculus of pseudo-differential operators. In fact, in [21, 23], Gröchenig and Heil prove that each operator with symbol in $M^{\infty,1}$ is continuous on all modulation spaces $M^{p,q}$, $p, q \in [1, \infty]$. In

particular, since $M^{2,2} = L^2$ it follows that any such operator is L^2 continuous, which was proved by Sjöstrand already in [36].

Some further generalizations to operators with symbols in more general modulation spaces were obtained in [24, 45, 46, 48]. Modulation spaces in pseudodifferential calculus is currently an active field of research (see e. g. [24, 28, 29, 38–40, 42, 45, 48]).

In the Weyl calculus of pseudo-differential operators, operator composition corresponds on the symbol level to the Weyl product, sometimes also called the twisted product, denoted by #. A problem in this field is to find conditions on the weight functions ω_j and $p_j, q_j \in [1, \infty]$, that are necessary and sufficient for the map

$$\mathscr{S}(\mathbf{R}^{2d}) \times \mathscr{S}(\mathbf{R}^{2d}) \ni (a_1, a_2) \mapsto a_1 \# a_2 \in \mathscr{S}(\mathbf{R}^{2d}) \tag{0.5}$$

to be uniquely extendable to a map from $M^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times M^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$ to $M^{p_0,q_0}_{(\omega_0)}(\mathbf{R}^{2d})$, which is continuous in the sense that for some constant $C > 0$ it holds

$$\|a_1 \# a_2\|_{M^{p_0,q_0}_{(\omega_0)}} \le C \|a_1\|_{M^{p_1,q_1}_{(\omega_1)}} \|a_2\|_{M^{p_2,q_2}_{(\omega_2)}}, \tag{0.6}$$

when $a_1 \in M^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d})$ and $a_2 \in M^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$. Important contributions in this context can be found in [22, 26, 29, 36, 42], where Theorem 0.3′ in [26] seems to be the most general result so far.

The Weyl product on the Fourier transform side is given by a twisted convolution, $*_\sigma$. It follows that the continuity questions here above are the same as finding appropriate conditions on ω_j and $p_j, q_j \in [1, \infty]$, in order for the map

$$\mathscr{S}(\mathbf{R}^{2d}) \times \mathscr{S}(\mathbf{R}^{2d}) \ni (a_1, a_2) \mapsto a_1 *_\sigma a_2 \in \mathscr{S}(\mathbf{R}^{2d}) \tag{0.7}$$

to be uniquely extendable to a map from $W^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times W^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$ to $W^{p_0,q_0}_{(\omega_0)}(\mathbf{R}^{2d})$, which is continuous in the sense that for some constant $C > 0$ it holds

$$\|a_1 *_\sigma a_2\|_{W^{p_0,q_0}_{(\omega_0)}} \le C \|a_1\|_{W^{p_1,q_1}_{(\omega_1)}} \|a_2\|_{W^{p_2,q_2}_{(\omega_2)}}, \tag{0.8}$$

when $a_1 \in W^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d})$ and $a_2 \in W^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$. In this context the continuity result which corresponds to Theorem 0.3′ in [26] is Theorem 2.3 in Section 2.

In the end of Section 2 we especially consider the case when $p_j = q_j = 2$. In this case, $W^{2,2}_{(\omega_j)}$ agrees with $L^2_{(\omega_j)}$, for appropriate choices of ω_j. Hence, for such ω_j, it follows immediately from Theorem 2.3 that the map (0.7) extends to a continuous mapping from $L^2_{(\omega_1)}(\mathbf{R}^{2d}) \times L^2_{(\omega_2)}(\mathbf{R}^{2d})$ to $L^2_{(\omega_0)}(\mathbf{R}^{2d})$, and that

$$\|a_1 *_\sigma a_2\|_{L^2_{(\omega_0)}} \le C \|a_1\|_{L^2_{(\omega_1)}} \|a_2\|_{L^2_{(\omega_2)}},$$

when $a_1 \in L^2_{(\omega_1)}(\mathbf{R}^{2d})$ and $a_2 \in L^2_{(\omega_2)}(\mathbf{R}^{2d})$. In Section 2 we prove a more general result, by combining this result with Young's inequality,

and then using interpolation. Finally we use these results in Section 3 to extend the class of possible window functions in the definition of modulation space norm.

In Sections 2–3, the second part of the paper, we cosider spaces of the form $s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)$ and $s_p^A(\mathscr{H}_1,\mathscr{H}_2)$, which consist of all $a\in\mathscr{S}'(\mathbf{R}^{2d})$ such that $a_t(x,D)$ and Aa respectively are Schatten-von Neumann operator of order $p\in[1,\infty]$ from the Hilbert space $\mathscr{H}_1$ to the Hilbert space $\mathscr{H}_2$. From the definitions it follows that the general continuity results for the usual Schatten-von Neumann classes carry over to the $s_{t,p}$ and s_p^A spaces, after the usual operator compositions have been replaced by the Weyl product and twisted convolution respectively.

In Section 4 we consider the case when $\mathscr{H}_j$ are equal to $M^2_{(\omega_j)}(\mathbf{R}^d)$ for some appropriate weight functions ω_j. In this situation we establish Young type result for dilated convolution and multiplication. More precisely, assume that

$$p_1{}^{-1}+p_2{}^{-1}=1+r^{-1},\qquad 1\le p_1,p_2,r\le\infty, \tag{0.9}$$

and that $\mathscr{H}_j=M^2_{(\omega_j)}$ and $\mathscr{H}_{k,j}=M^2_{(\omega_{k,j})}$ are appropriate. If $t_1,t_2\neq 0$ and $c_1t_1^2+c_2t_2^2=1$, for some choice of $c_1,c_2\in\{\pm1\}$, then we prove that

$$a_1(\,\cdot\,/t_1)*a_2(\,\cdot\,/t_2)\in s_{t,r}(\mathscr{H}_1,\mathscr{H}_2),\quad\text{when}\quad a_j\in s_{t,p_j}(\mathscr{H}_{1,j},\mathscr{H}_{2,j}),$$

and

$$a_1(t_1\,\cdot\,)\cdot a_2(t_2\,\cdot\,)\in s_{t,r}(\mathscr{H}_1,\mathscr{H}_2),\quad\text{when}\quad a_j\in s_{t,p_j}(\mathscr{H}_{1,j},\mathscr{H}_{2,j}).$$

In particular, if $\mathscr{H}_j=\mathscr{H}_{k,j}=L^2(\mathbf{R}^d)$, then we recover the results in Section 3 in [43]. Furthermore, if each of the operators $a_j^w(x,D)$ are positive with respect to the L^2 form, then we prove that the same is true for $(a_1(\,\cdot\,/t_1)*a_2(\,\cdot\,/t_2))^w(x,D)$.

1. Preliminaries

In this section we recall some notations and basic results. The proofs are in general omitted.

We start by discussing appropriate conditions for the involved weight functions. Assume that ω and v are positive and measureable functions on $\mathbf{R}^d$. Then ω is called v-moderate if

$$\omega(x+y)\le C\omega(x)v(y) \tag{1.1}$$

for some constant C which is independent of $x,y\in\mathbf{R}^d$. If v in (1.1) can be chosen as a polynomial, then ω is called polynomially moderated. We let $\mathscr{P}(\mathbf{R}^d)$ be the set of all polynomially moderated functions on $\mathbf{R}^d$. If $\omega(x,\xi)\in\mathscr{P}(\mathbf{R}^{2d})$ is constant with respect to the x-variable (ξ-variable), then we sometimes write $\omega(\xi)$ ($\omega(x)$) instead of $\omega(x,\xi)$. In this case we consider ω as an element in $\mathscr{P}(\mathbf{R}^{2d})$ or in $\mathscr{P}(\mathbf{R}^d)$ depending on the situation.

We recall that the Fourier transform $\mathscr{F}$ on $\mathscr{S}'(\mathbf{R}^d)$, which takes the form (0.2) when $f \in L^1(\mathbf{R}^d)$, is a homeomorphism on $\mathscr{S}'(\mathbf{R}^d)$ which restricts to a homeomorphism on $\mathscr{S}(\mathbf{R}^d)$ and to a unitary operator on $L^2(\mathbf{R}^d)$.

Let $\varphi \in \mathscr{S}'(\mathbf{R}^d)$ be fixed, and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Then the short-time Fourier transform $V_\varphi f(x,\xi)$ of f with respect to the *window function* φ is the tempered distribution on $\mathbf{R}^{2d}$ which is defined by

$$V_\varphi f(x,\xi) \equiv \mathscr{F}(f\,\overline{\varphi(\,\cdot\,-x)})(\xi).$$

If $f,\varphi \in \mathscr{S}(\mathbf{R}^d)$, then it follows that

$$V_\varphi f(x,\xi) = (2\pi)^{-d/2}\int f(y)\overline{\varphi(y-x)}e^{-i\langle y,\xi\rangle}\,dy.$$

Next we recall some properties on modulation spaces and their Fourier transforms. Assume that $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ and that $p,q \in [1,\infty]$. Then the mixed Lebesgue space $L^{p,q}_{1,(\omega)}(\mathbf{R}^{2d})$ consists of all $F \in L^1_{loc}(\mathbf{R}^{2d})$ such that $\|F\|_{L^{p,q}_{1,(\omega)}} < \infty$, and $L^{p,q}_{2,(\omega)}(\mathbf{R}^{2d})$ consists of all $F \in L^1_{loc}(\mathbf{R}^{2d})$ such that $\|F\|_{L^{p,q}_{2,(\omega)}} < \infty$. Here

$$\|F\|_{L^{p,q}_{1,(\omega)}} = \Big(\int\Big(\int |F(x,\xi)\omega(x,\xi)|^p\,dx\Big)^{q/p}\,d\xi\Big)^{1/q},$$

and

$$\|F\|_{L^{p,q}_{2,(\omega)}} = \Big(\int\Big(\int |F(x,\xi)\omega(x,\xi)|^q\,d\xi\Big)^{p/q}\,dx\Big)^{1/p},$$

with obvious modifications when $p=\infty$ or $q=\infty$.

Assume that $p,q\in[1,\infty]$, $\omega\in\mathscr{P}(\mathbf{R}^{2d})$ and $\varphi\in\mathscr{S}(\mathbf{R}^d)\setminus 0$ are fixed. Then the *modulation spaces* $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbf{R}^d)$ are the Banach spaces which consist of all $f\in\mathscr{S}'(\mathbf{R}^d)$ such that

$$\|f\|_{M^{p,q}_{(\omega)}} \equiv \|V_\varphi f\|_{L^{p,q}_{1,(\omega)}} < \infty, \text{ and } \|f\|_{W^{p,q}_{(\omega)}} \equiv \|V_\varphi f\|_{L^{p,q}_{2,(\omega)}} < \infty \tag{1.2}$$

respectively. The definitions of $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbf{R}^d)$ are independent of the choice of φ and different φ gives rise to equivalent norms. (See Proposition 1.1 below). From the fact that

$$V_{\widehat\varphi}\widehat f(\xi,-x) = e^{i\langle x,\xi\rangle}V_{\check\varphi}f(x,\xi),\qquad \check\varphi(x)=\varphi(-x),$$

it follows that

$$f\in W^{q,p}_{(\omega)}(\mathbf{R}^d)\quad\Longleftrightarrow\quad \widehat f\in M^{p,q}_{(\omega_0)}(\mathbf{R}^d),\qquad \omega_0(\xi,-x)=\omega(x,\xi).$$

For conveniency we set $M^p_{(\omega)} = M^{p,p}_{(\omega)}$, which agrees with $W^p_{(\omega)} = W^{p,p}_{(\omega)}$. Furthermore we set $M^{p,q} = M^{p,q}_{(\omega)}$ and $W^{p,q} = W^{p,q}_{(\omega)}$ if $\omega\equiv 1$. If ω is given by $\omega(x,\xi)=\omega_1(x)\omega_2(\xi)$, for some $\omega_1,\omega_2\in\mathscr{P}(\mathbf{R}^d)$, then $W^{p,q}_{(\omega)}$ is a Wiener amalgam space, introduced by Feichtinger in [8].

The proof of the following proposition is omitted, since the results can be found in [9, 10, 14–16, 21, 45–48]. Here and in what follows, $p'\in$

$[1,\infty]$ denotes the conjugate exponent of $p\in[1,\infty]$, i. e. $1/p+1/p'=1$ should be fulfilled.

Proposition 1.1. *Assume that $p,q,p_j,q_j\in[1,\infty]$ for $j=1,2$, and $\omega,\omega_1,\omega_2,v\in\mathscr P(\mathbf R^{2d})$ are such that ω is v-moderate and $\omega_2\le C\omega_1$ for some constant $C>0$. Then the following are true:*

(1) *if $\varphi\in M^1_{(v)}(\mathbf R^d)\backslash 0$, then $f\in M^{p,q}_{(\omega)}(\mathbf R^d)$ if and only if (1.2) holds, i. e. $M^{p,q}_{(\omega)}(\mathbf R^d)$. Moreover, $M^{p,q}_{(\omega)}$ is a Banach space under the norm in (1.2) and different choices of φ give rise to equivalent norms;*

(2) *if $p_1\le p_2$ and $q_1\le q_2$ then*

$$\mathscr S(\mathbf R^d)\hookrightarrow M^{p_1,q_1}_{(\omega_1)}(\mathbf R^n)\hookrightarrow M^{p_2,q_2}_{(\omega_2)}(\mathbf R^d)\hookrightarrow \mathscr S'(\mathbf R^d);$$

(3) *the L^2 product $(\,\cdot\,,\,\cdot\,)$ on $\mathscr S$ extends to a continuous map from $M^{p,q}_{(\omega)}(\mathbf R^n)\times M^{p',q'}_{(1/\omega)}(\mathbf R^d)$ to $\mathbf C$. On the other hand, if $\|a\|=\sup|(a,b)|$, where the supremum is taken over all $b\in\mathscr S(\mathbf R^d)$ such that $\|b\|_{M^{p',q'}_{(1/\omega)}}\le 1$, then $\|\cdot\|$ and $\|\cdot\|_{M^{p,q}_{(\omega)}}$ are equivalent norms;*

(4) *if $p,q<\infty$, then $\mathscr S(\mathbf R^d)$ is dense in $M^{p,q}_{(\omega)}(\mathbf R^d)$ and the dual space of $M^{p,q}_{(\omega)}(\mathbf R^d)$ can be identified with $M^{p',q'}_{(1/\omega)}(\mathbf R^d)$, through the form $(\,\cdot\,,\,\cdot\,)_{L^2}$. Moreover, $\mathscr S(\mathbf R^d)$ is weakly dense in $M^{\infty}_{(\omega)}(\mathbf R^d)$.*

Similar facts hold if the $M^{p,q}_{(\omega)}$ spaces are replaced by $W^{p,q}_{(\omega)}$ spaces.

Proposition 1.1 (1) allows us be rather vague concerning the choice of $\varphi\in M^1_{(v)}\setminus 0$ in (1.2). For example, if $C>0$ is a constant and $\mathscr A$ is a subset of $\mathscr S'$, then $\|a\|_{M^{p,q}_{(\omega)}}\le C$ for every $a\in\mathscr A$, means that the inequality holds for some choice of $\varphi\in M^1_{(v)}\setminus 0$ and every $a\in\mathscr A$. Evidently, a similar inequality is true for any other choice of $\varphi\in M^1_{(v)}\setminus 0$, with a suitable constant, larger than C if necessary.

In the following remark we list some other properties for modulation spaces. Here and in what follows we let $\langle x\rangle=(1+|x|^2)^{1/2}$, when $x\in\mathbf R^d$.

Remark 1.2. Assume that $p,p_1,p_2,q,q_1,q_2\in[1,\infty]$ are such that

$$q_1\le\min(p,p'),\quad q_2\ge\max(p,p'),\quad p_1\le\min(q,q'),\quad p_2\ge\max(q,q'),$$

and that $\omega,v\in\mathscr P(\mathbf R^{2d})$ are such that ω is v-moderate. Then the following is true:

(1) if $p\le q$, then $W^{p,q}_{(\omega)}(\mathbf R^d)\subseteq M^{p,q}_{(\omega)}(\mathbf R^d)$, and if $p\ge q$, then $M^{p,q}_{(\omega)}(\mathbf R^d)\subseteq W^{p,q}_{(\omega)}(\mathbf R^d)$. Furthermore, if $\omega(x,\xi)=\omega(x)$, then

$$M^{p,q_1}_{(\omega)}(\mathbf R^d)\subseteq W^{p,q_1}_{(\omega)}(\mathbf R^d)\subseteq L^p_{(\omega)}(\mathbf R^d)\subseteq W^{p,q_2}_{(\omega)}(\mathbf R^d)\subseteq M^{p,q_2}_{(\omega)}(\mathbf R^d).$$

In particular, $M^2_{(\omega)} = W^2_{(\omega)} = L^2_{(\omega)}$. If instead $\omega(x,\xi) = \omega(\xi)$, then

$$W^{p_1,q}_{(\omega)}(\mathbf{R}^d) \subseteq M^{p,q_1}_{(\omega)}(\mathbf{R}^d) \subseteq \mathscr{F}L^q_{(\omega)}(\mathbf{R}^d) \subseteq M^{p_2,q}_{(\omega)}(\mathbf{R}^d) \subseteq W^{p_2,q}_{(\omega)}(\mathbf{R}^d).$$

Here $\mathscr{F}L^q_{(\omega_0)}(\mathbf{R}^d)$ consists of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that

$$\|\widehat{f}\,\omega_0\|_{L^q} < \infty;$$

(2) if $\omega(x,\xi) = \omega(x)$, then

$$M^{p,q}_{(\omega)}(\mathbf{R}^d) \subseteq C(\mathbf{R}^d) \quad \iff \quad W^{p,q}_{(\omega)}(\mathbf{R}^d) \subseteq C(\mathbf{R}^d) \quad \iff \quad q = 1.$$

(3) $M^{1,\infty}(\mathbf{R}^d)$ and $W^{1,\infty}(\mathbf{R}^d)$ are convolution algebras. If $C'_B(\mathbf{R}^d)$ is the set of all measures on $\mathbf{R}^d$ with bounded mass, then

$$C'_B(\mathbf{R}^d) \subseteq W^{1,\infty}(\mathbf{R}^d) \subseteq M^{1,\infty}(\mathbf{R}^d);$$

(4) if $x_0 \in \mathbf{R}^d$ is fixed and $\omega_0(\xi) = \omega(x_0,\xi)$, then

$$M^{p,q}_{(\omega)} \cap \mathscr{E}' = W^{p,q}_{(\omega)} \cap \mathscr{E}' = \mathscr{F}L^q_{(\omega_0)} \cap \mathscr{E}';$$

(5) for each $x,\xi \in \mathbf{R}^d$ and modulation space norm $\|\cdot\|$ we have

$$\|e^{i\langle \,\cdot\, ,\xi\rangle} f(\,\cdot - x)\| \le Cv(x,\xi)\|f\|,$$

for some constant C which is independent of $f \in \mathscr{S}'(\mathbf{R}^d)$;

(6) if $\tilde{\omega}(x,\xi) = \omega(x,-\xi)$ then $f \in M^{p,q}_{(\omega)}$ if and only if $\overline{f} \in M^{p,q}_{(\tilde{\omega})}$;

(7) if $s \in \mathbf{R}$ and $\omega(x,\xi) = \langle \xi \rangle^s$, then $M^2_{(\omega)} = W^2_{(\omega)}$ agrees with the Sobolev space H^2_s, which consists of all $f \in \mathscr{S}'$ such that $\mathscr{F}^{-1}(\langle\,\cdot\,\rangle^s \widehat{f}) \in L^2$.

(See e. g. [9, 10, 14–16, 21, 45–48].)

Remark 1.3. Assume that $s,t \in \mathbf{R}$. In many applications it is common that functions of the form

$$\sigma_s(x) = \langle x \rangle^s \quad \text{and} \quad \sigma_{s,t}(x,\xi) \equiv \langle x \rangle^t \langle \xi \rangle^s,$$

are involved. Then it easilly follows that σ_s and $\sigma_{s,t}$ are $\sigma_{|s|}$-moderate and $\sigma_{|s|,|t|}$-moderate respectively. For convenience we set

$$M^{p,q}_{s,t} = M^{p,q}_{(\sigma_{s,t})} \quad M^{p,q}_s = M^{p,q}_{(\sigma_s)},$$

and similarily for other function and distribution spaces, e. g. we set $L^p_s = L^p_{(\sigma_s)}$. We note that for such weight functions we have

$$M^{p,q}_{s,t}(\mathbf{R}^d) = \{\, f \in \mathscr{S}'(\mathbf{R}^d)\,;\, \langle x \rangle^t \langle D \rangle^s f \in M^{p,q}(\mathbf{R}^d)\,\}, \quad s,t \in \mathbf{R}$$

and

$$M^{p,q}_s(\mathbf{R}^d) = M^{p,q}_{s,0}(\mathbf{R}^d) \cap M^{p,q}_{0,s}(\mathbf{R}^d), \quad s \ge 0.$$

(Cf. [46].) In particular, since $M^2 = L^2$, it holds $M^2_s = H^2_s \cap L^2_s$, when $s \ge 0$. (See also [21, 23].)

Next we recall some facts in Chapter XVIII in [27] concerning pseudo-differential operators. Assume that $a \in \mathscr{S}(\mathbf{R}^{2d})$, and that $t \in \mathbf{R}$ is fixed. Then the pseudo-differential operator $a_t(x,D)$ in (0.1) is a linear and continuous operator on $\mathscr{S}(\mathbf{R}^d)$. For general $a \in \mathscr{S}'(\mathbf{R}^{2d})$, the pseudo-differential operator $a_t(x,D)$ is defined as the continuous operator from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ with distribution kernel given by (0.3). This definition makes sense, since the mappings $\mathscr{F}_2$ and $F(x,y) \mapsto F((1-t)x+ty, y-x)$ are homeomorphisms on $\mathscr{S}'(\mathbf{R}^{2d})$. Furthermore, by Schwartz kernel theorem it follows that the map $a \mapsto a_t(x,D)$ is a bijection from $\mathscr{S}'(\mathbf{R}^{2d})$ to the set of linear and continuous operators from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$.

We recall that for $s,t \in \mathbf{R}$ and $a,b \in \mathscr{S}'(\mathbf{R}^{2d})$, we have

$$a_s(x,D) = b_t(x,D) \qquad \Longleftrightarrow \qquad b(x,\xi) = e^{i(t-s)\langle D_x,D_\xi\rangle}a(x,\xi). \tag{1.3}$$

(Note here that the right-hand side makes sense, since $e^{i(t-s)\langle D_\xi,D_x\rangle}$ on the Fourier transform side is a multiplication by the bounded function $e^{i(t-s)\langle x,\xi\rangle}$.

Assume that $t \in \mathbf{R}$ and $a \in \mathscr{S}'(\mathbf{R}^{2d})$ are fixed. Then a is called a rank-one element with respect to t, if the corresponding pseudo-differential operator is of rank-one, i. e.

$$a_t(x,D)f = (f,f_2)f_1, \tag{1.4}$$

for some $f_1,f_2 \in \mathscr{S}'(\mathbf{R}^d)$. By straight-forward computations it follows that (1.4) is fulfilled, if and only if $a = (2\pi)^{d/2}W^t_{f_1,f_2}$, where the $W^t_{f_1,f_2}$ t-Wigner distribution, defined by the formula

$$W^t_{f_1,f_2}(x,\xi) \equiv \mathscr{F}(f_1(x+t\,\cdot\,)\overline{f_2(x-(1-t)\,\cdot\,)})(\xi), \tag{1.5}$$

which takes the form

$$W^t_{f_1,f_2}(x,\xi) = (2\pi)^{-d/2}\int f_1(x+ty)\overline{f_2(x-(1-t)y)}e^{-i\langle y,\xi\rangle}\,dy,$$

when $f_1,f_2 \in \mathscr{S}(\mathbf{R}^d)$. By combining these facts with (1.3), it follows that

$$W^t_{f_1,f_2} = e^{i(t-s)\langle D_x,D_\xi\rangle}W^s_{f_1,f_2}, \tag{1.6}$$

for each $f_1,f_2 \in \mathscr{S}'(\mathbf{R}^d)$ and $s,t \in \mathbf{R}$. Since the Weyl case ($t=1/2$) is particulary important to us, we set $W^t_{f_1,f_2} = W_{f_1,f_2}$ when $t=1/2$. It follows that W_{f_1,f_2} is the usual (cross-) Wigner distribution of f_1 and f_2.

Next we recall the definition of symplectic Fourier transform, Weyl product, twisted convolution and related objects. Assume that $a,b \in \mathscr{S}'(\mathbf{R}^{2d})$. Then the Weyl product $a\#b$ between a and b is the function or distribution which fulfills $(a\#b)^w(x,D) = a^w(x,D)\circ b^w(x,D)$, provided the right-hand side makes sense. More general, if $t \in \mathbf{R}$, then the product $\#_t$ is defined by the formula

$$(a\#_t b)_t(x,D) = a_t(x,D)\circ b_t(x,D), \tag{1.7}$$

provided the right-hand side makes sense as a continuous operator from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$.

The even-dimensional vector space $\mathbf{R}^{2d}$ is a (real) symplectic vector space with the (standard) symplectic form

$$\sigma(X,Y) = \sigma\big((x,\xi);(y,\eta)\big) = \langle y,\xi\rangle - \langle x,\eta\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product on $\mathbf{R}^d$.

The symplectic Fourier transform for $a \in \mathscr{S}(\mathbf{R}^{2d})$ is defined by the formula

$$(\mathscr{F}_\sigma a)(X) = \pi^{-d}\int a(Y)e^{2i\sigma(X,Y)}\,dY.$$

Then $\mathscr{F}_\sigma^{-1} = \mathscr{F}_\sigma$ is continuous on $\mathscr{S}(\mathbf{R}^{2d})$, and extends as usual to a homeomorphism on $\mathscr{S}'(\mathbf{R}^{2d})$, and to a unitary map on $L^2(\mathbf{R}^{2d})$. The symplectic short-time Fourier transform of $a \in \mathscr{S}'(\mathbf{R}^{2d})$ with respect to the window function $\varphi \in \mathscr{S}'(\mathbf{R}^{2d})$ is defined by

$$\mathcal{V}_\varphi a(X,Y) = \mathscr{F}_\sigma\big(a\,\varphi(\,\cdot\,-X)\big)(Y),\quad X,Y\in\mathbf{R}^{2d}.$$

Assume that $\omega \in \mathscr{P}(\mathbf{R}^{4d})$. Then we let $\mathcal{M}^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ and $\mathcal{W}^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ denote the modulation spaces, where the symplectic short-time Fourier transform is used instead of the usual short-time Fourier transform in the definitions of the norms. It follows that any property valid for $M^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ or $W^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ carry over to $\mathcal{M}^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ and $\mathcal{W}^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ respectively. For example, for the symplectic short-time Fourier transform we have

$$\mathcal{V}_{\mathscr{F}_\sigma\varphi}(\mathscr{F}_\sigma a)(X,Y) = e^{2i\sigma(Y,X)}\mathcal{V}_\varphi a(Y,X), \tag{1.8}$$

which implies that

$$\mathscr{F}_\sigma\mathcal{M}^{p,q}_{(\omega)}(\mathbf{R}^{2d}) = \mathcal{W}^{q,p}_{(\omega_0)}(\mathbf{R}^{2d}),\qquad \omega_0(X,Y) = \omega(Y,X). \tag{1.9}$$

Assume that $a,b\in\mathscr{S}(\mathbf{R}^{2d})$. Then the twisted convolution of a and b is defined by the formula

$$(a *_\sigma b)(X) = (2/\pi)^{d/2}\int a(X-Y)b(Y)e^{2i\sigma(X,Y)}\,dY. \tag{1.10}$$

The definition of $*_\sigma$ extends in different ways. For example, it extends to a continuous multiplication on $L^p(\mathbf{R}^{2d})$ when $p\in[1,2]$, and to a continuous map from $\mathscr{S}'(\mathbf{R}^{2d})\times\mathscr{S}(\mathbf{R}^{2d})$ to $\mathscr{S}'(\mathbf{R}^{2d})$. If $a,b\in\mathscr{S}'(\mathbf{R}^{2d})$, then $a\#b$ makes sense if and only if $a *_\sigma \widehat{b}$ makes sense, and then

$$a\#b = (2\pi)^{-d/2}a *_\sigma (\mathscr{F}_\sigma b). \tag{1.11}$$

We also remark that for the twisted convolution we have

$$\mathscr{F}_\sigma(a *_\sigma b) = (\mathscr{F}_\sigma a) *_\sigma b = \check{a} *_\sigma (\mathscr{F}_\sigma b), \tag{1.12}$$

where $\check{a}(X) = a(-X)$ (cf. [41, 43, 44]). A combination of (1.11) and (1.12) give

$$\mathscr{F}_\sigma(a\#b) = (2\pi)^{-d/2}(\mathscr{F}_\sigma a) *_\sigma (\mathscr{F}_\sigma b). \tag{1.13}$$

Next we consider the operator A in (0.4). We recall that A is a homeomorphism on $\mathscr{S}'(\mathbf{R}^{2d})$, which restricts to a homeomorphism on $\mathscr{S}(\mathbf{R}^{2n})$ and to unitary map on $L^2(\mathbf{R}^{2d})$. Furthermore, by Fourier's inversion formula it follows that the inverse is given by

$$(A^{-1}U)(x,\xi) = \mathscr{F}(U(\,\cdot\,/2 - x,\,\cdot\,/2 + x))(-\xi)$$

when $U \in \mathscr{S}'(\mathbf{R}^{2d})$, which takes the form of

$$(A^{-1}U)(x,\xi) = (2\pi)^{-d/2} \int e^{i\langle y,\xi\rangle} U(y/2 - x, y/2 + x)\, dy,$$

when $U \in \mathscr{S}(\mathbf{R}^{2d})$.

An important reason for considering the operator Aa is its close connection with the Weyl calculus (cf. Lemma 1.4), and that

$$A(a *_\sigma b) = Aa \circ Ab, \tag{1.14}$$

when $a \in \mathscr{S}'(\mathbf{R}^{2d})$ ($a \in \mathscr{S}(\mathbf{R}^{2d})$) and $b \in \mathscr{S}(\mathbf{R}^{2d})$ ($b \in \mathscr{S}'(\mathbf{R}^{2d})$). (See [17,41,43,44].) Here we have identified operators with their kernels.

In the following lemma we list some facts about the operator A. The result is a consequence of Fourier's inversion formula, and the verifications are left for the reader.

Lemma 1.4. *Let A be as above and let $U = Aa$ where $a \in \mathscr{S}'(\mathbf{R}^{2d})$. Then the following is true:*

(1) $\check U = A\check a$, *if* $\check a(X) = a(-X)$;

(2) $J_{\mathscr{F}}U = A\mathscr{F}_\sigma a$, *where* $J_{\mathscr{F}}U(x,y) = U(-x,y)$;

(3) $A(\mathscr{F}_\sigma a) = (2\pi)^{d/2}a^w(x,D)$ *and* $(a^w(x,D)f,g) = (2\pi)^{-d/2}(Aa, \check g \otimes \overline{f})$ *when* $f,g \in \mathscr{S}(\mathbf{R}^d)$;

(4) *the Hilbert space adjoint of Aa equals $A\widetilde a$, where $\widetilde a(X) = \overline{a(-X)}$. Furthermore, if $a_1, a_2, b \in \mathscr{S}(\mathbf{R}^{2d})$, then*

$$(a_1 *_\sigma a_2, b) = (a_1, b *_\sigma \widetilde a_2) = (a_2, \widetilde a_1 *_\sigma b), \quad (a_1 *_\sigma a_2) *_\sigma b = a_1 *_\sigma (a_2 *_\sigma b).$$

Next we recall some facts on operators which are positive with respect to the L^2 form (see the end of the introduction). Assume that T is a linear and continuous operator from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^n)$. We say that T is positive semi-definite and write $T \ge 0$, if $(Tf,f)_{L^2} \ge 0$ for every $f \in \mathscr{S}(\mathbf{R}^d)$. Furthermore, we also consider distributions which are positive with respect to the twisted convolution.

Definition 1.5. Assume that $a \in \mathscr{D}'(\mathbf{R}^{2d})$. Then a is called a σ-positive distribution if $(a *_\sigma \varphi, \varphi)_{L^2} \ge 0$ for all $\varphi \in C_0^\infty(\mathbf{R}^{2d})$. The set of all σ-positive distributions is denoted by $\mathscr{S}'_+(\mathbf{R}^{2d})$. Furthermore, $\mathscr{S}'_+(\mathbf{R}^{2d}) \cap C(\mathbf{R}^{2d})$, the set of σ-positive (continuous) functions, is denoted by $C_+(\mathbf{R}^{2d})$.

In [44] that it is proved that

$$\mathscr{S}'_+ \subseteq \mathscr{S}, \quad \text{and} \quad C_+ \subseteq L^2 \cap C_B \cap \mathscr{F}C_B,$$

where $C_B(\mathbf{R}^{2d})$ is the set of bounded continuous functions on $\mathbf{R}^d$ which turns to zero at infinity.

The following result is a straight-forward consequence of the definitions, and shows that positivity in the sense of Definition 1.5 is closely related to positive in operator and pseudo-differential operator theory. (See also [41, 43].)

Proposition 1.6. *Assume that $a \in \mathscr{S}'(\mathbf{R}^{2d})$. Then*

$$a \in \mathscr{S}'_+(\mathbf{R}^{2d}) \quad \Longleftrightarrow \quad Aa \ge 0 \quad \Longleftrightarrow \quad (\mathscr{F}_\sigma a)^w(x,D) \ge 0.$$

In the end of Section 5 we also consider Toeplitz operators. Assume that $a \in \mathscr{S}(\mathbf{R}^{2d})$, $h_1, h_2 \in \mathscr{S}(\mathbf{R}^d)$, and set $\check f(x) = f(-x)$ when f is a distribution. Then the Toeplitz operator $\operatorname{Tp}_{h_1,h_2}(a)$, with symbol a, and window functions h_1 and h_2, is defined by the formula

$$(\operatorname{Tp}_{h_1,h_2}(a)f_1, f_2) = (aV_{\check h_1}f_1, V_{\check h_2}f_2) = (a(2\,\cdot\,)W_{f_1,h_1}, W_{f_2,h_2})$$

when $f_1, f_2 \in \mathscr{S}(\mathbf{R}^d)$. The definition of $\operatorname{Tp}_{h_1,h_2}(a)$ extends in several ways. For example, several extensions are presented in [5, 25, 41, 43, 45, 48, 49], in such ways that h_1, h_2 and a are permitted to belong to Lebesgue spaces, modulation spaces or Schatten-von Neumann symbol classes.

The most of these extensions are based on the fact that the pseudo-differential operator symbol of a Toeplitz opertor can be expressed by the relation

$$\begin{aligned} &(a * u)_t(x,D) = \operatorname{Tp}_{h_1,h_2}(a) \quad \text{with} \\ &\qquad u(X) = (2\pi)^{-d/2} W^t_{h_2,h_1}(-X), \end{aligned} \tag{1.15}$$

when $t = 1/2$, h_1, h_2 are suitable window functions on $\mathbf{R}^d$ and a is an appropriate distribution on $\mathbf{R}^{2d}$. (See e. g. [5, 41, 43, 45–47, 49].) For general t, (1.15) is an immediate consequence of the case $t = 1/2$, (1.6), and the fact that

$$e^{i(t-s)\langle D_x, D_\xi\rangle}(a * u) = a * (e^{i(t-s)\langle D_x, D_\xi\rangle} u),$$

which follows by integrations by parts.

2. Twisted convolution on modulation spaces and Lebesgue spaces

In this section we discuss algebraic properties of the twisted convolution when acting on modulation spaces of the form $\mathcal{W}^{p,q}_{(\omega)}$. The most general result is equivalent to Theorem 0.3′ in [26], which concerns continuity for the Weyl product on modulation spaces of the form $\mathcal{M}^{p,q}_{(\omega)}$. Thereafter we use this result to establish continuity properties for the twisted convolution when acting on weighted Lebesgue spaces.

The following lemma is important in our investigations. The proof is omitted since the result is an immediate consequence of Lemma 4.4 in [42] and its proof, (1.8), (1.11) and (1.12).

Lemma 2.1. *Assume that $a_1 \in \mathscr{S}'(\mathbf{R}^{2d})$, $a_2 \in \mathscr{S}(\mathbf{R}^{2d})$ and $\varphi_1, \varphi_2 \in \mathscr{S}(\mathbf{R}^{2d})$. Then the following is true:*

(1) *if $\varphi = \pi^d \varphi_1 \# \varphi_2$, then $\varphi \in \mathscr{S}(\mathbf{R}^{2d})$, the map*

$$Z \mapsto e^{2i\sigma(Z,Y)}(\mathcal{V}_{\chi_1}a_1)(X-Y+Z,Z)\,(\mathcal{V}_{\chi_2}a_2)(X+Z,Y-Z)$$

belongs to $L^1(\mathbf{R}^{2d})$, and

$$\begin{aligned}\tag{2.1} &\mathcal{V}_\varphi(a_1\#a_2)(X,Y)\\ &\quad = \int e^{2i\sigma(Z,Y)}(\mathcal{V}_{\chi_1}a_1)(X-Y+Z,Z)\,(\mathcal{V}_{\chi_2}a_2)(X+Z,Y-Z)\,dZ;\end{aligned}$$

(2) *if $\varphi = 2^{-d}\varphi_1 *_\sigma \varphi_2$, then $\varphi \in \mathscr{S}(\mathbf{R}^{2d})$, the map*

$$Z \mapsto e^{2i\sigma(X,Z-Y)}(\mathcal{V}_{\chi_1}a_1)(X-Y+Z,Z)\,(\mathcal{V}_{\chi_2}a_2)(Y-Z,X+Z)$$

belongs to $L^1(\mathbf{R}^{2d})$, and

$$\begin{aligned}\tag{2.2} &\mathcal{V}_\varphi(a_1 *_\sigma a_2)(X,Y)\\ &\quad = \int e^{2i\sigma(X,Z-Y)}(\mathcal{V}_{\chi_1}a_1)(X-Y+Z,Z)\,(\mathcal{V}_{\chi_2}a_2)(Y-Z,X+Z)\,dZ\end{aligned}$$

The first part of the latter result is used in [26] to prove the following result, which is essentially a restatement of Theorem 0.3′ in [26]. Here we assume that the involved weight functions satisfy

$$\omega_0(X,Y) \le C\omega_1(X-Y+Z,Z)\omega_2(X+Z,Y-Z), \quad X,Y,Z \in \mathbf{R}^{2d}. \tag{2.3}$$

for some constant $C>0$, and that $p_j, q_j \in [1,\infty]$ satisfy

$$\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_0} = 1-\Big(\frac{1}{q_1}+\frac{1}{q_2}-\frac{1}{q_0}\Big) \tag{2.4}$$

and

$$0 \le \frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_0} \le \frac{1}{p_j}, \frac{1}{q_j} \le \frac{1}{q_1}+\frac{1}{q_2}-\frac{1}{q_0}, \quad j=0,1,2. \tag{2.5}$$

Theorem 2.2. *Assume that $\omega_0, \omega_1, \omega_2 \in \mathscr{P}(\mathbf{R}^{4d})$ satisfy (2.3), and that $p_j, q_j \in [1,\infty]$ for $j = 0,1,2$, satisfy (2.4) and (2.5). Then the map (0.5) on $\mathscr{S}(\mathbf{R}^{2d})$ extends uniquely to a continuous map from $\mathcal{M}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times \mathcal{M}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$ to $\mathcal{M}^{p_0,q_0}_{(\omega_0)}(\mathbf{R}^{2d})$, and for some constant $C>0$, the bound (0.6) holds for every $a_1 \in \mathcal{M}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d})$ and $a_2 \in \mathcal{M}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$.*

The next result is an immediate consequence of (1.9), (1.13) and Theorem 2.2. Here the condition (2.3) should be replaced by

$$\omega_0(X,Y) \le C\omega_1(X-Y+Z,Z)\omega_2(Y-Z,X+Z), \quad X,Y,Z \in \mathbf{R}^{2d}. \tag{2.6}$$

and the condition (2.5) should be replaced by

$$0 \le \frac{1}{q_1}+\frac{1}{q_2}-\frac{1}{q_0} \le \frac{1}{p_j}, \frac{1}{q_j} \le \frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_0}, \quad j=0,1,2. \tag{2.7}$$

Theorem 2.3. *Assume that $\omega_0,\omega_1,\omega_2 \in \mathscr{P}(\mathbf{R}^{4d})$ satisfy (2.6), and that $p_j,q_j \in [1,\infty]$ for $j=0,1,2$, satisfy (2.4) and (2.7). Then the map (0.7) on $\mathscr{S}(\mathbf{R}^{2d})$ extends uniquely to a continuous map from $\mathcal{W}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times \mathcal{W}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$ to $\mathcal{W}^{p_0,q_0}_{(\omega_0)}(\mathbf{R}^{2d})$, and for some constant $C>0$, the bound (0.8) holds for every $a_1 \in \mathcal{W}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d})$ and $a_2 \in \mathcal{W}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$.*

By using Theorem 2.3 we may generalize Proposition 1.4 in [43] to involve continuity of the twisted convolution on weighted Lebesgue spaces. Here the condition (2.6) is replaced by

$$\omega_0(X_1+X_2) \le C\omega_1(X_1)\omega_2(X_2) \tag{2.8}$$

Theorem 2.4. *Assume that $\omega_0,\omega_1,\omega_2 \in \mathscr{P}(\mathbf{R}^{2d})$ and $p,p_1,p_2 \in [1,\infty]$ satisfy (2.8), $p_1,p_2 \le p$ and*

$$\max\left(\frac{1}{p},\frac{1}{p'}\right) \le \frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p} \le 1,$$

for some constant C. Then the map (0.7) extends uniquely to a continuous mapping from $L^{p_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times L^{p_2}_{(\omega_2)}(\mathbf{R}^{2d})$ to $L^{p}_{(\omega_0)}(\mathbf{R}^{2d})$. Furthermore, for some constant C it holds

$$\|a_1 *_\sigma a_2\|_{L^p_{(\omega_0)}} \le C\|a_1\|_{L^{p_1}_{(\omega_1)}}\|a_2\|_{L^{p_2}_{(\omega_2)}},$$
$$\text{when} \quad a_1 \in L^{p_1}_{(\omega_1)}(\mathbf{R}^{2d}), \quad \text{and} \quad a_2 \in L^{p_2}_{(\omega_2)}(\mathbf{R}^{2d}).$$

Proof. From the assumptions it follows that at most one of p_1 and p_2 are equal to ∞. By reasons of symmetry we may therefore assume that $p_2<\infty$.

Since $W^2_{(\omega)} = M^2_{(\omega)} = L^2_{(\omega)}$ when $\omega(X,Y)=\omega(X)$, in view of Theorem 2.2 in [46], the result follows from Theorem 2.3 in the case $p_1 = p_2 = p = 2$.

Now assume that $1/p_1+1/p_2-1/p=1$, $a_1 \in L^{p_1}(\mathbf{R}^{2d})$ and that $a_2 \in \mathscr{S}(\mathbf{R}^{2d})$. Then

$$\|a_1 *_\sigma a_2\|_{L^p_{(\omega_0)}} \le (2/\pi)^{d/2}\|\,|a_1|*|a_2|\,\|_{L^p_{(\omega_0)}} \le C\|a_1\|_{L^{p_1}_{(\omega_1)}}\|a_2\|_{L^{p_2}_{(\omega_2)}},$$

by Young's inequality. The result now follows in this case from the fact that $\mathscr{S}$ is dense in $L^{p_2}_{(\omega_2)}$, when $p_2<\infty$.

The result now follows in the general case by multi-linear interpolation between the case $p_1=p_2=p=2$ and the case $1/p_1+1/p_2-1/p=1$, using Theorem 4.4.1 in [1] and the fact that

$$(L^{p_1}_{(\omega)}(\mathbf{R}^{2d}),(L^{p_2}_{(\omega)}(\mathbf{R}^{2d}))_{[\theta]} = L^{p_0}_{(\omega)}(\mathbf{R}^{2d}), \quad \text{when} \quad \frac{1-\theta}{p_1}+\frac{\theta}{p_2}=\frac{1}{p_0}.$$

(Cf. Chapter 5 in [1].) The proof is complete. □

By letting $p_1 = p$ and $p_2 = q \le \min(p,p')$, or $p_2 = p$ and $p_1 = q \le \min(p,p')$, Theorem 2.4 takes the following form:

Corollary 2.5. *Assume that $\omega_0,\omega_1,\omega_2 \in \mathscr P(\mathbf R^{2d})$ and $p,q\in[1,\infty]$ satisfy* (2.8)*, and $q \le \min(p,p')$ for some constant C. Then the map* (0.7) *extends uniquely to a continuous mapping from $L^p_{(\omega_1)}(\mathbf R^{2d})\times L^q_{(\omega_2)}(\mathbf R^{2d})$ or $L^q_{(\omega_1)}(\mathbf R^{2d})\times L^p_{(\omega_2)}(\mathbf R^{2d})$ to $L^p_{(\omega_0)}(\mathbf R^{2d})$.*

In the next section we need the following refinement of Theorem 2.4 concerning mixed Lebesgue spaces.

Theorem 2.4′. *Assume that $k\in\{1,2\}$, $\omega_0,\omega_1,\omega_2 \in \mathscr P(\mathbf R^{2d})$ and $p,p_j,q,q_j\in[1,\infty]$ for $j=1,2$ satisfy* (2.8)*, $p_1,p_2\le p$, $q_1,q_2\le q$ and*

$$\max\Big(\frac1p,\frac1{p'},\frac1q,\frac1{q'}\Big)\le \frac1{p_1}+\frac1{p_2}-\frac1p\, ,\ \frac1{q_1}+\frac1{q_2}-\frac1q\le 1,$$

for some constant C. Then the map (0.7) *extends uniquely to a continuous mapping from $L^{p_1,q_1}_{k,(\omega_1)}(\mathbf R^{2d})\times L^{p_2,q_2}_{k,(\omega_2)}(\mathbf R^{2d})$ to $L^{p,q}_{k,(\omega_0)}(\mathbf R^{2d})$. Furthermore, for some constant C it holds*

$$\|a_1 *_\sigma a_2\|_{L^{p,q}_{k,(\omega_0)}} \le C\|a_1\|_{L^{p_1,q_1}_{k,(\omega_1)}}\|a_2\|_{L^{p_2,q_2}_{k,(\omega_2)}},$$
$$\text{when}\quad a_1\in L^{p_1,q_1}_{k,(\omega_1)}(\mathbf R^{2d}),\quad \text{and}\quad a_2\in L^{p_2,q_2}_{k,(\omega_2)}(\mathbf R^{2d}).$$

Proof. The result follows from Minkowski's inequality when $p_1=q_1=1$ and when $p_2=q_2=1$. Furthermore, the result follows in the case $p_1=p_2=q_1=q_2=2$ from Theorem 2.4. In the general case, the result follows from these cases and multi-linear interpolation. $\square$

3. Window functions in modulation space norms

In this section we use the results in the previous section to prove that the class of permitted windows in the modulation space norms can be extended. More precisely we have the following.

Proposition 3.1. *Assume that $p,p_0,q,q_0\in[1,\infty]$ and $\omega,v\in\mathscr P(\mathbf R^{2d})$ are such that $p_0,q_0\le\min(p,p',q,q')$, $\check v = v$ and ω is v-moderate. Also assume that $f\in\mathscr S'(\mathbf R^d)$. Then the following is true:*

1. *if $\varphi\in M^{p_0,q_0}_{(v)}(\mathbf R^d)\setminus 0$, then $f\in M^{p,q}_{(\omega)}(\mathbf R^d)$ if and only if $V_\varphi f\in L^{p,q}_{1,(\omega)}(\mathbf R^{2d})$. Furthermore, $\|f\|\equiv\|V_\varphi f\|_{L^{p,q}_{1,(\omega)}}$ defines a norm for $M^{p,q}_{(\omega)}(\mathbf R^d)$, and different choices of φ give rise to equivalent norms;*
2. *if $\varphi\in W^{p_0,q_0}_{(v)}(\mathbf R^d)\setminus 0$, then $f\in W^{p,q}_{(\omega)}(\mathbf R^d)$ if and only if $V_\varphi f\in L^{p,q}_{2,(\omega)}(\mathbf R^{2d})$. Furthermore, $\|f\|\equiv\|V_\varphi f\|_{L^{p,q}_{2,(\omega)}}$ defines a norm for $W^{p,q}_{(\omega)}(\mathbf R^d)$, and different choices of φ give rise to equivalent norms.*

For the proof we note that the relation between Wigner distributions (cf. (1.5) with $t = 1/2$) and short-time Fourier is given by

$$W_{f,g}(x,\xi) = 2^d e^{i\langle x,\xi\rangle/2} V_{\check g} f(2x,2\xi),$$

which implies that

$$\|W_{f,\check\varphi}\|_{L^{p,q}_{k,(\omega_0)}} = 2^d \|V_\varphi f\|_{L^{p,q}_{k,(\omega)}}, \quad \text{when} \quad \omega_0(x,\xi) = \omega(2x,2\xi) \tag{3.1}$$

for $k = 1,2$.

Finally, by Fourier's inversion formula it follows that if $f_1, g_2 \in \mathscr S'(\mathbf R^d)$ and $f_1, g_2 \in L^2(\mathbf R^d)$, then

$$W_{f_1,g_1} *_\sigma W_{f_2,g_2} = (\check f_2, g_1)_{L^2} W_{f_1,g_2}. \tag{3.2}$$

Proof of Theorem 3.1. We may assume that $p_0 = q_0 = \min(p,p',q,q')$. Assume that $\varphi,\psi \in M^{p_0,q_0}_{(v)}(\mathbf R^d) \subseteq L^2(\mathbf R^d)$, where the inclusion follows from the fact that $p_0,q_0 \le 2$ and $v \ge c$ for some constant $c>0$. Since $\check v = v$, and $\|V_\varphi \psi\|_{L^{p_0,q_0}_{k,(v)}} = \|V_\psi \varphi\|_{L^{p_0,q_0}_{k,(v)}}$ when $\check v = v$, the result follows if we prove that

$$\|V_\varphi f\|_{L^{p,q}_{k,(\omega)}} \le C \|V_\psi f\|_{L^{p,q}_{k,(\omega)}} \|V_\varphi \psi\|_{L^{p_0,q_0}_{k,(v)}}, \tag{3.3}$$

for some constant C which is independent of $f \in \mathscr S'(\mathbf R^d)$ and $\varphi,\psi \in M^{p_0,q_0}_{(v)}(\mathbf R^d)$.

If $p_1 = p$, $p_2 = p_0$, $q_1 = q$, $q_2 = q_0$, $\omega_0 = \omega(2\,\cdot\,)$ and $v_0 = v(2\,\cdot\,)$, then Theorem 2.4$'$ and (3.2) give

$$\begin{aligned}
\|V_\varphi f\|_{L^{p,q}_{k,(\omega)}} &= C_1 \|W_{f,\check\varphi}\|_{L^{p,q}_{k,(\omega_0)}} \\
&= C_2 \|W_{f,\check\psi} *_\sigma W_{\psi,\check\varphi}\|_{L^{p,q}_{k,(\omega_0)}} \le C_3 \|W_{f,\check\psi}\|_{L^{p,q}_{k,(\omega_0)}} \|W_{\psi,\check\varphi}\|_{L^{p_0,q_0}_{k,(v_0)}} \\
&= C_4 \|V_\psi f\|_{L^{p,q}_{k,(\omega)}} \|V_\varphi \psi\|_{L^{p_0,q_0}_{k,(v)}},
\end{aligned}$$

and (3.3) follows. The proof is complete. □

4. Schatten-von Neumann classes and pseudo-differential operators

In this section we discuss Schatten-von Neumann classes of pseudo-differential operators from a Hilbert space $\mathscr H_1$ to another Hilbert space $\mathscr H_2$. Schatten-von Neumann classes were introduced by R. Schatten in [33] in the case $\mathscr H_1 = \mathscr H_2$. (See also [35]). The general situation, when $\mathscr H_1$ is not necessarily equal to $\mathscr H_2$, has thereafter been considered in [2, 34, 50].

Let $\mathrm{ON}(\mathscr H_j)$, $j = 1,2$, denote the family of orthonormal sequences in $\mathscr H_j$, and assume that $T : \mathscr H_1 \to \mathscr H_2$ is linear, and that $p \in [1,\infty]$. Then set

$$\|T\|_{\mathscr I_p} = \|T\|_{\mathscr I_p(\mathscr H_1,\mathscr H_2)} \equiv \sup \Big(\sum |(Tf_j, g_j)_{\mathscr H_2}|^p \Big)^{1/p}$$

(with obvious modifications when $p = \infty$). Here the supremum is taken over all $(f_j) \in \mathrm{ON}(\mathscr{H}_1)$ and $(g_j) \in \mathrm{ON}(\mathscr{H}_2)$. Then $\mathscr{I}_p = \mathscr{I}_p(\mathscr{H}_1, \mathscr{H}_2)$, the Schatten-von Neumann class of order p, consists of all linear and continuous operators T from $\mathscr{H}_1$ to $\mathscr{H}_2$ such that $\|T\|_{\mathscr{I}_p(\mathscr{H}_1,\mathscr{H}_2)}$ is finite. We note that $\mathscr{I}_\infty(\mathscr{H}_1, \mathscr{H}_2)$ agrees with $\mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$, the set of linear and continuous operators from $\mathscr{H}_1$ to $\mathscr{H}_2$, with equality in norms. We also let $\mathcal{K}(\mathscr{H}_1, \mathscr{H}_2)$ be the set of all linear and compact operators from $\mathscr{H}_1$ to $\mathscr{H}_2$, and equip this space with the operator norm as usual. (Note that the notation $\mathscr{I}_\sharp(\mathscr{H}_1, \mathscr{H}_2)$ was used instead of $\mathcal{K}(\mathscr{H}_1, \mathscr{H}_2)$ in [48].) If $\mathscr{H}_1 = \mathscr{H}_2$, then the shorter notation $\mathscr{I}_p(\mathscr{H}_1)$ is used instead of $\mathscr{I}_p(\mathscr{H}_1, \mathscr{H}_2)$, and similarily for $\mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$ and $\mathcal{K}(\mathscr{H}_1, \mathscr{H}_2)$.

Assume that (e_j) is an orthonormal basis in $\mathscr{H}_1$, and that $S \in \mathscr{I}_1(\mathscr{H}_1)$. Then the trace of S is defined as

$$\operatorname{tr}_{\mathscr{H}_1} S = \sum (Se_j, e_j)_{\mathscr{H}_1}.$$

For each pairs of operators $T_1, T_2 \in \mathscr{I}_\infty(\mathscr{H}_1, \mathscr{H}_2)$ such that $T_2^* \circ T_1 \in \mathscr{I}_1(\mathscr{H}_1)$, the sesqui-linear form

$$(T_1, T_2) = (T_1, T_2)_{\mathscr{H}_1, \mathscr{H}_2} \equiv \operatorname{tr}_{\mathscr{H}_1}(T_2^* \circ T_1)$$

of T_1 and T_2 is well-defined. Here we note that $T \in \mathscr{I}_p(\mathscr{H}_1, \mathscr{H}_2)$ if and only if $T^* \in \mathscr{I}_p(\mathscr{H}_2, \mathscr{H}_1)$. We refer to [2, 35, 50] for more facts about Schatten-von Neumann classes.

In order for discussing Schatten-von Neumann operators within the theory of pseudo-differential operators, we assume from now on that the Hilbert spaces $\mathscr{H}, \mathscr{H}_0, \mathscr{H}_1, \mathscr{H}_2, \dots$ are "tempered" in the following sense.

Definition 4.1. The Hilbert space $\mathscr{H} \subseteq \mathscr{S}'(\mathbf{R}^d)$ is called *tempered* (on $\mathbf{R}^d$), if $\mathscr{S}(\mathbf{R}^d)$ is contained and dense in $\mathscr{H}$.

Assume that $\mathscr{H}$ is a tempered Hilbert space on $\mathbf{R}^d$. Then we let $\check{\mathscr{H}}$ and $\mathscr{H}^\tau$ be the sets of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that $\check f \in \mathscr{H}$ and $\overline{f} \in \mathscr{H}$ respectively. Then $\check{\mathscr{H}}$ and $\mathscr{H}^\tau$ are tempered Hilbert spaces under the norms

$$\|f\|_{\check{\mathscr{H}}} \equiv \|\check f\|_{\mathscr{H}} \quad \text{and} \quad \|f\|_{\mathscr{H}^\tau} \equiv \|\overline{f}\|_{\mathscr{H}}$$

respectively.

The L^2*-dual*, $\mathscr{H}'$, of $\mathscr{H}$ is the set of all $\varphi \in \mathscr{S}'(\mathbf{R}^d)$ such that

$$\|\varphi\|_{\mathscr{H}'} \equiv \sup |(\varphi, f)_{L^2(\mathbf{R}^d)}|$$

is finite. Here the supremum is taken over all $f \in \mathscr{S}(\mathbf{R}^d)$ such that $\|f\|_{\mathscr{H}} \le 1$. Assume that $\varphi \in \mathscr{H}'$. Since $\mathscr{S}$ is dense in $\mathscr{H}$, it follows from the definitions that the map $f \mapsto (\varphi, f)_{L^2}$ from $\mathscr{S}(\mathbf{R}^d)$ to $\mathbf{C}$ extends uniquely to a continuous mapping from $\mathscr{H}$ to $\mathbf{C}$. The following version of Riesz lemma is useful for us. In order to be self-contained, we also give a proof.

Lemma 4.2. *Assume that $\mathscr{H} \subseteq \mathscr{S}'(\mathbf{R}^d)$ is a tempered Hilbert space with L^2-dual $\mathscr{H}'$. Then the following is true:*

(1) *$\mathscr{H}'$ is a tempered Hilbert space which can be identified with the dual space of $\mathscr{H}$ through the L^2-form;*

(2) *there is a unique map $T_{\mathscr{H}}$ from $\mathscr{H}$ to $\mathscr{H}'$ such that*

$$(f,g)_{\mathscr{H}} = (T_{\mathscr{H}}f,g)_{L^2(\mathbf{R}^d)}; \tag{4.1}$$

(3) *if $T_{\mathscr{H}}$ is the map in (2), $(e_j)_{j\in I}$ is an orthonormal basis in $\mathscr{H}$ and $\varepsilon_j = T_{\mathscr{H}}e_j$, then $T_{\mathscr{H}}$ is isometric, $(\varepsilon_j)_{j\in I}$ is an orthonormal basis and*

$$(\varepsilon_j, e_k)_{L^2(\mathbf{R}^d)} = \delta_{j,k}.$$

Proof. We have that $\mathscr{S} \subseteq \mathscr{H}' \subseteq \mathscr{S}'$, and since $\mathscr{S}$ is dense in $\mathscr{H}$, it follows that $\mathscr{S}$ is dense also in $\mathscr{H}'$.

First assume that $f \in \mathscr{H}$, $g \in \mathscr{S}(\mathbf{R}^d)$, and let $T_{\mathscr{H}}f$ in $\mathscr{S}'(\mathbf{R}^d)$ be defined by (4.1). By the definitions it follow that $T_{\mathscr{H}}f \in \mathscr{H}'$, and that $T_{\mathscr{H}}$ from $\mathscr{H}$ to $\mathscr{H}'$ is isometric. Furthermore, since the dual space of $\mathscr{H}$ can be identified with itself, under the scalar product of $\mathscr{H}$, the asserted duality properties of $\mathscr{H}'$ follow.

Let $(e_j)_{j\in I}$ be an arbitrary orthonormal basis in $\mathscr{H}$, and let $\varepsilon_j = T_{\mathscr{H}}e_j$. Then it follows that $\|\varepsilon_j\|_{\mathscr{H}'} = 1$ and

$$(\varepsilon_j, e_k)_{L^2} = (e_j, e_k)_{\mathscr{H}} = \delta_{j,k}.$$

Furthermore, if

$$f = \sum \alpha_j e_j, \qquad \varphi = \sum \alpha_j \varepsilon_j$$

$$g = \sum \beta_j e_j, \qquad \gamma = \sum \beta_j \varepsilon_j$$

are finite sums, and we set $(\varphi,\gamma)_{\mathscr{H}'} \equiv (f,g)_{\mathscr{H}}$, then it follows that $(\,\cdot\,,\,\cdot\,)_{\mathscr{H}'}$ defines a scalar product on such finite sums in $\mathscr{H}'$, and that $\|\varphi\|^2_{\mathscr{H}'} = (\varphi,\varphi)_{\mathscr{H}'}$. By continuity extensions it now follows that $(\varphi,\gamma)_{\mathscr{H}'}$ extends uniquely to each $\varphi,\gamma \in \mathscr{H}'$, and that the identity $\|\varphi\|^2_{\mathscr{H}'} = (\varphi,\varphi)_{\mathscr{H}'}$ holds. This proves the result. □

In what follows we call the basis (ε_j) in Lemma 4.2 as the *dual basis* of (e_j).

Corollary 4.3. *Assume that $\mathscr{H}$ is a tempered Hilbert space on $\mathbf{R}^d$. Then*

$$M^2_{s,s}(\mathbf{R}^d) \subseteq \mathscr{H}, \mathscr{H}' \subseteq M^2_{-s,-s}(\mathbf{R}^d),$$

for some $s \geq 0$. Furthermore, $M^2_{s,s}(\mathbf{R}^d)$ is dense in $\mathscr{H}$ and $\mathscr{H}'$, which in turn are dense in $M^2_{-s,-s}(\mathbf{R}^d)$.

Proof. The topology in $\mathscr{S}$ can be obtained by using the semi-norms

$$\|f\|_{[s]} \equiv \sum_{|\alpha|,|\beta|\leq s} \|x^\alpha D^\beta f\|_{L^2}, \quad s = 0,1,2,\dots .$$

From the fact that $\mathscr{S}$ is continuously embedded in $\mathscr{H}$ and in $\mathscr{H}'$, it therefore follows that

$$\|f\|_{\mathscr{H}} \le C\|f\|_{[s]} \quad \text{and} \quad \|\varphi\|_{\mathscr{H}'} \le C\|\varphi\|_{[s]},$$

when $f \in \mathscr{S}$, provided s is chosen large enough.

Since the completion of $\mathscr{S}(\mathbf{R}^d)$ under $\|\cdot\|_{[s]}$ is equal to $M^2_{s,s}(\mathbf{R}^d)$, the result follows by a standard argument of approximation, using the duality propoerties in Proposition1.1 (4), together with the facts that $\mathscr{S}$ is dense in $\mathscr{H}$, $\mathscr{H}'$, $M^2_{s,s}$ and in $M^2_{-s,-s}$. The proof is complete. $\square$

Assume that $\mathscr{H}_1, \mathscr{H}_2 \subseteq \mathscr{S}'(\mathbf{R}^d)$ are tempered Hilbert spaces, $t \in \mathbf{R}$ is fixed and that $p \in [1,\infty]$. Then we let $s^A_p(\mathscr{H}_1,\mathscr{H}_2)$ and $s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)$ be the sets of all $a \in \mathscr{S}'(\mathbf{R}^{2d})$ such that $Aa \in \mathscr{I}_p(\mathscr{H}_1,\mathscr{H}_2)$ and $a_t(x,D) \in \mathscr{I}_p(\mathscr{H}_1,\mathscr{H}_2)$ respectively. We also let $s^A_\sharp(\mathscr{H}_1,\mathscr{H}_2)$ and $s_{t,\sharp}(\mathscr{H}_1,\mathscr{H}_2)$ be the set of all $a \in \mathscr{S}'(\mathbf{R}^{2d})$ such that $Aa \in \mathcal{K}(\mathscr{H}_1,\mathscr{H}_2)$ and $a_t(x,D) \in \mathcal{K}(\mathscr{H}_1,\mathscr{H}_2)$ respectively. These spaces are equipped by the norms

$$\|a\|_{s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)} \equiv \|a_t(x,D)\|_{\mathscr{I}_p(\mathscr{H}_1,\mathscr{H}_2)}, \qquad \|a\|_{s^A_p(\mathscr{H}_1,\mathscr{H}_2)} \equiv \|Aa\|_{\mathscr{I}_p(\mathscr{H}_1,\mathscr{H}_2)},$$

$$\|a\|_{s_{t,\sharp}(\mathscr{H}_1,\mathscr{H}_2)} \equiv \|a\|_{s_{t,\infty}(\mathscr{H}_1,\mathscr{H}_2)}, \qquad \|a\|_{s^A_\sharp(\mathscr{H}_1,\mathscr{H}_2)} \equiv \|a\|_{s^A_\infty(\mathscr{H}_1,\mathscr{H}_2)}.$$

Since the mappings $a \mapsto Aa$ and $a \mapsto a_t(x,D)$ are bijections from $\mathscr{S}'(\mathbf{R}^{2d})$ to the set of linear and continuous operators from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$, it follows that $a \mapsto Aa$ and $a \mapsto a_t(x,D)$ restrict to isometric bijections from $s^A_p(\mathscr{H}_1,\mathscr{H}_2)$ and $s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)$ respectively to $\mathscr{I}_p(\mathscr{H}_1,\mathscr{H}_2)$. Consequently, the properties for $\mathscr{I}_p(\mathscr{H}_1,\mathscr{H}_2)$ carry over to $s^A_p(\mathscr{H}_1,\mathscr{H}_2)$ and $s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)$. In particular, elements in $s^A_1(\mathscr{H}_1,\mathscr{H}_2)$ of finite rank (i. e. elements of the form $a \in s^A_1(\mathscr{H}_1,\mathscr{H}_2)$ such that Aa is a finite rank operator) are dense in $s^A_\sharp(\mathscr{H}_1,\mathscr{H}_2)$ and in $s^A_p(\mathscr{H}_1,\mathscr{H}_2)$ when $p < \infty$. Since the Weyl quantization is particularly important in our considerations we also set

$$s^w_p = s_{t,p} \quad \text{and} \quad s^w_\sharp = s_{t,\sharp}, \quad \text{when} \quad t = 1/2.$$

If $\omega_1, \omega_2 \in \mathscr{P}(\mathbf{R}^{2d})$, then we use the notation $s^A_p(\omega_1,\omega_2)$ instead of $s^A_p(M^2_{(\omega_1)}, M^2_{(\omega_2)})$. Furthermore we set $s^A_p(\omega_1,\omega_2) = s^A_p(\mathbf{R}^{2d})$ if in addition $\omega_1 = \omega_2 = 1$. In the same way the notations for $s_{t,p}$, s^w_p, $s_{t,\sharp}$ and $s^w_\sharp$ are simplified.

Remark 4.4. Assume that $t, t_1, t_2 \in \mathbf{R}$, $p \in [1,\infty]$, $\mathscr{H}_1, \mathscr{H}_2$ are tempered Hilbert spaces on $\mathbf{R}^d$ and that $a, b \in \mathscr{S}'(\mathbf{R}^{2d})$. Then it follows by Fourier's inversion formula that the map $e^{it\langle D_x, D_\xi\rangle}$ is a homeomorphism on $\mathscr{S}(\mathbf{R}^{2d})$ which extends uniquely to a homeomorphism on $\mathscr{S}'(\mathbf{R}^{2d})$. Furthermore, by (1.3) it follows that $e^{i(t_2-t_1)\langle D_x, D_\xi\rangle}$ restricts to an isometric bijection from $s_{t_1,p}(\mathscr{H}_1,\mathscr{H}_2)$ to $s_{t_2,p}(\mathscr{H}_1,\mathscr{H}_2)$.

The following proposition shows how $s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)$, $s^A_p(\mathscr{H}_1,\mathscr{H}_2)$ and other similar spaces are linked together. The proof is essentially the

same as the proof of Proposition 5.1 in [50]. Here and in what follows we let $a^\tau(x,\xi) = \overline{a(x,-\xi)}$ be the "torsion" of $a \in \mathscr S'(\mathbf R^{2d})$.

Proposition 4.5. *Assume that $t \in \mathbf R$, $\mathscr H_1, \mathscr H_2$ are tempered Hilbert spaces in $\mathbf R^d$, $a \in \mathscr S'(\mathbf R^{2d})$, and that $p \in [1,\infty]$. Then $s_p^w(\mathscr H_1,\mathscr H_2) = s_p^A(\mathscr H_1,\check{\mathscr H}_2)$. Furthermore, the following conditions are equivalent:*

(1) $a \in s_p^w(\mathscr H_1,\mathscr H_2)$;

(2) $\mathscr F_\sigma a \in s_p^w(\mathscr H_1,\check{\mathscr H}_2) = s_p^A(\mathscr H_1,\mathscr H_2)$;

(3) $\overline a \in s_p^w(\mathscr H_2',\mathscr H_1')$;

(4) $a^\tau \in s_p^A(\mathscr H_1^\tau,\mathscr H_2^\tau)$;

(5) $\check a \in s_p^w(\check{\mathscr H}_1,\check{\mathscr H}_2)$;

(6) $\widetilde a \in s_p^w(\check{\mathscr H}_2',\check{\mathscr H}_1')$;

(7) $e^{i(t-1/2)\langle D_\xi,D_x\rangle}a \in s_{t,p}(\mathscr H_1,\mathscr H_2)$.

Proof. Let $a_1 = \mathscr F_\sigma a$, $a_2 = \overline a$, $a_3 = a^\tau$, $a_4 = \check a$ and $a_5 = \widetilde a$. Then the equivalences follow immediately from Remark 4.4 and the equalities

$$
\begin{aligned}
(a^w(x,D)f,g) &= (a_1^w(x,D)f,\check g) = (f,a_2^w(x,D)g)\\
&= \overline{(a_3^w(x,D)\overline f,\overline g)} = (a_4^w(x,D)\check f,\check g) = (\check f,a_5^w(x,D)\check g),
\end{aligned}
$$

when $a \in \mathscr S'(\mathbf R^{2d})$ and $f,g \in \mathscr S(\mathbf R^d)$. Here the first equality follows from the fact that if $K(x,y)$ is the distribution kernel of $a^w(x,D)$, then $K(-x,y)$ is the distribution kernel of $(\mathscr F_\sigma a)^w(x,D) = (2\pi)^{-d/2}Aa$. (Cf. [41,43].) The proof is complete. □

In Remarks 4.6 and 4.7 below we list some properties which follow from well-known results in the theory of Schatten-von Neumann classes in combination with (1.7), (1.14) and the fact that the mappings $a \mapsto a_t(x,D)$ and $a \mapsto Aa$ are isometric bijections from $s_{t,p}(\mathscr H_1,\mathscr H_2)$ and $s_p^w(\mathscr H_1,\mathscr H_2)$ respectively to $\mathscr I_p(\mathscr H_1,\mathscr H_2)$. (We refer to [2,35,50] for corresponding results about the $\mathscr I_p$ spaces.) Here the forms $(\,\cdot\,,\,\cdot\,)_{s_{t,2}(\mathscr H_1,\mathscr H_2)}$ and $(\,\cdot\,,\,\cdot\,)_{s_2^A(\mathscr H_1,\mathscr H_2)}$ are defined by the formula

$$
(a,b)_{s_{t,2}(\mathscr H_1,\mathscr H_2)} = (a_t(x,D),b_t(x,D))_{\mathscr I_2(\mathscr H_1,\mathscr H_2)},\quad a,b \in s_{t,2}(\mathscr H_1,\mathscr H_2)
$$

and

$$
(a,b)_{s_2^A(\mathscr H_1,\mathscr H_2)} = (Aa,Ab)_{\mathscr I_2(\mathscr H_1,\mathscr H_2)},\qquad a,b \in s_2^A(\mathscr H_1,\mathscr H_2).
$$

We also recall that $p' \in [1,\infty]$ is the conjugate exponent for $p \in [1,\infty]$, i.e. $1/p+1/p' = 1$. Finally, the set l_0^∞ consists of all sequences in l^∞ which turns to zero at infinity, and l_0^1 consists of all sequences $(\lambda_j)_{j\in I}$ such that $\lambda_j = 0$ except for finite numbers of $j \in I$.

Remark 4.6. Assume that $p,p_j,q,r \in [1,\infty]$ for $1 \le j \le 2$, $t \in \mathbf R$, and that $\mathscr H_1,\dots,\mathscr H_4$ are tempered Hilbert spaces on $\mathbf R^d$. Then the following is true:

(1) the set $s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)$ is a Banach space which increases with the parameter p. If in addition $p<\infty$ and $p_1\le p_2$, then $s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)\subseteq s_{t,\sharp}(\mathscr{H}_1,\mathscr{H}_2)$, $s_{t,1}(\mathscr{H}_1,\mathscr{H}_2)$ is dense in $s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)$ and in $s_{t,\sharp}(\mathscr{H}_1,\mathscr{H}_2)$, and

$$\|a\|_{s_{t,p_2}(\mathscr{H}_1,\mathscr{H}_2)}\le \|a\|_{s_{t,p_1}(\mathscr{H}_1,\mathscr{H}_2)},\quad a\in s_{t,\infty}(\mathscr{H}_1,\mathscr{H}_2); \tag{4.2}$$

(2) equality in (4.2) is attained, if and only if a is of rank one, and then $\|a\|_{s_{t,p(\mathscr{H}_1,\mathscr{H}_2)_p}}=(2\pi)^{-d/2}\|f_0\|_{\mathscr{H}_1}\|g_0\|_{\mathscr{H}_2}$, when a is given by (1.5);

(3) if $1/p_1+1/p_2=1/r$, $a_1\in s_{t,p_1}(\mathscr{H}_1,\mathscr{H}_2)$ and $a_1\in s_{t,p_1}(\mathscr{H}_2,\mathscr{H}_3)$, then $a_2\#_t a_1\in s_{t,r}(\mathscr{H}_1,\mathscr{H}_3)$, and

$$\|a_2\#_t a_1\|_{s_{t,r}(\mathscr{H}_1,\mathscr{H}_3)}\le \|a_1\|_{s_{t,p_1}(\mathscr{H}_1,\mathscr{H}_2)}\|a_2\|_{s_{t,p_2}(\mathscr{H}_2,\mathscr{H}_3)}. \tag{4.3}$$

On the other hand, for any $a\in s_{t,r}(\mathscr{H}_1,\mathscr{H}_3)$, there are elements $a_1\in s_{t,p_1}(\mathscr{H}_1,\mathscr{H}_2)$ and $a_2\in s_{t,p_2}(\mathscr{H}_2,\mathscr{H}_3)$ such that $a=a_2\#_t a_1$ and equality holds in (4.3);

(4) if $\mathscr{H}_1\subseteq\mathscr{H}_2$ and $\mathscr{H}_3\subseteq\mathscr{H}_4$, then $s_{t,p}(\mathscr{H}_2,\mathscr{H}_3)\subseteq s_{t,p}(\mathscr{H}_1,\mathscr{H}_4)$.

Similar facts hold when the $s_{t,p}$ spaces and the product $\#_t$ are replaced by s_p^A spaces and $*_\sigma$.

Remark 4.7. Assume that $p,p_j,q,r\in[1,\infty]$ for $1\le j\le 2$, $t\in\mathbf{R}$, and that $\mathscr{H}_1,\dots,\mathscr{H}_4$ are tempered Hilbert spaces on $\mathbf{R}^d$. Then the following is true:

(1) the form $(\,\cdot\,,\,\cdot\,)_{s_{t,2}(\mathscr{H}_1,\mathscr{H}_2)}$ on $s_{t,1}(\mathscr{H}_1,\mathscr{H}_2)$ extends uniquely to a sesqui-linear and continuous form from $s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)\times s_{t,p'}(\mathscr{H}_1,\mathscr{H}_2)$ to $\mathbf{C}$, and for every $a_1\in s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)$ and $a_2\in s_{t,p'}(\mathscr{H}_1,\mathscr{H}_2)$, it holds

$$(a_1,a_2)_{s_{t,2}(\mathscr{H}_1,\mathscr{H}_2)}=\overline{(a_2,a_1)_{s_{t,2}(\mathscr{H}_1,\mathscr{H}_2)}},$$

$$|(a_1,a_2)_{s_{t,2}(\mathscr{H}_1,\mathscr{H}_2)}|\le \|a_1\|_{s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)}\|a_2\|_{s_{t,p'}(\mathscr{H}_1,\mathscr{H}_2)}\quad\text{and}$$

$$\|a_1\|_{s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)}=\sup|(a_1,b)_{s_{t,2}(\mathscr{H}_1,\mathscr{H}_2)}|,$$

where the supremum is taken over all $b\in s_{t,p'}(\mathscr{H}_1,\mathscr{H}_2)$ such that $\|b\|_{s_{t,p'}(\mathscr{H}_1,\mathscr{H}_2)}\le 1$. If in addition $p<\infty$, then the dual space of $s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)$ can be identified with $s_{t,p'}(\mathscr{H}_1,\mathscr{H}_2)$ through this form;

(2) if $a\in s_{t,\sharp}(\mathscr{H}_1,\mathscr{H}_2)$, then

$$a_t(x,D)f=\sum_{j=1}^\infty \lambda_j(f,f_j)_{\mathscr{H}_1}g_j, \tag{4.4}$$

holds for some $(f_j)_{j=1}^\infty\in\mathrm{ON}(\mathscr{H}_1)$, $(g_j)_{j=1}^\infty\in\mathrm{ON}(\mathscr{H}_2)$ and $\lambda=(\lambda_j)_{j=1}^\infty\in l_0^\infty$, where the operator on the right-hand side of (4.4)

convergences with respect to the operator norm. Moreover, $a \in s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$, if and only if $\lambda \in l^p$, and then

$$\|a\|_{s_{t,p}} = (2\pi)^{-d/2}\|\lambda\|_{l^p}$$

and the operator on the right-hand side of (4.4) converges with respect to the norm $\|\cdot\|_{s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)}$;

(3) If $0 \le \theta \le 1$ is such that $1/p = (1-\theta)/p_1 + \theta/p_2$, then the (complex) interpolation space

$$(s_{t,p_1}(\mathscr{H}_1, \mathscr{H}_2), s_{t,p_2}(\mathscr{H}_1, \mathscr{H}_2))_{[\theta]} = s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$$

with equality in norms.

Similar facts hold when the $s_{t,p}$ spaces are replaced by s_p^A spaces.

A problem with the form $(\,\cdot\,,\,\cdot\,)_{s_{t,2}(\mathscr{H}_1,\mathscr{H}_2)}$ in Remark 4.7 is the somewhat complicated structure. In the following we show that there is a canonical way to replace this form with $(\,\cdot\,,\mathbf{C})_{L^2}$. We start with the following result concerning polar decomposition of compact operators.

Proposition 4.8. *Assume that $\mathscr{H}_1$ and $\mathscr{H}_2$ are tempered Hilbert spaces on $\mathbf{R}^d$, $a \in s_{t,\sharp}(\mathscr{H}_1, \mathscr{H}_2)$ and that $p \in [1,\infty]$. Then*

$$a \equiv \sum_{j\in I} \lambda_j W^t_{g_j,\varphi_j}$$

(with norm convergence) for some orthonormal sequences $(\varphi_j)_{j\in I}$ in $\mathscr{H}_1'$ and $(f_j)_{j\in I}$ in $\mathscr{H}_2$, and a sequence $(\lambda_j)_{j\in I} \in l_0^\infty$ of non-negative real numbers which decreases to zero at infinity. Furthermore, $a \in s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$, if and only if $(\lambda_j)_{j\in I} \in l^p$, and

$$\|a\|_{s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)} = (2\pi)^{-d/2}\|(\lambda_j)_{j\in I}\|_{l^p}.$$

Proof. By Remark 4.7 (2) it follows that if $f \in \mathscr{S}(\mathbf{R}^d)$, then

$$a_t(x,D)f(x) = \sum_{j\in I} \lambda_j (f, f_j)_{\mathscr{H}_1} g_j \tag{4.5}$$

for some orthonormal sequences (f_j) in $\mathscr{H}_1$ and (g_j) in $\mathscr{H}_2$, and a sequence $(\lambda_j) \in l_0^\infty$ of non-negative real numbers which decreases to zero at infinity. Now let $(\varphi_j)_{j\in I}$ be an orthonormal sequence in $\mathscr{H}_1'$ such that $(\varphi_j, g_k)_{L^2} = \delta_{j,k}$. Then $(f, f_j)_{\mathscr{H}_1} = (f, \gamma_j)_{L^2}$, and the result follows from (4.5), and the fact that

$$(W^t_{g_j,\varphi_j})_t(x,D)f = (f,\varphi_j)_{L^2} g_j = (f, f_j)_{\mathscr{H}_1} g_j.$$

The proof is complete. □

Next we prove that the duals for $s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$ and $s_p^A(\mathscr{H}_1, \mathscr{H}_2)$ can be identified with $s_{t,p'}(\mathscr{H}_1', \mathscr{H}_2')$ and $s_{p'}^A(\mathscr{H}_1', \mathscr{H}_2')$ respectively through the form $(\,\cdot\,,\mathbf{C})_{L^2}$.

Theorem 4.9. *Assume that $t \in \mathbf{R}$, $p \in [1,\infty)$ and that $\mathscr{H}_1, \mathscr{H}_2$ are tempered Hilbert spaces on $\mathbf{R}^d$. Then the L^2 form on $\mathscr{S}(\mathbf{R}^{2d})$ extends uniquely to a duality between $s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)$ and $s_{t,p'}(\mathscr{H}_1',\mathscr{H}_2')$, and the dual space for $s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)$ can be identified with $s_{t,p'}(\mathscr{H}_1',\mathscr{H}_2')$ through this form. Moreover, if $\ell \in s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)^*$ and $a \in s_{t,p'}(\mathscr{H}_1',\mathscr{H}_2')$ are such that $\overline{\ell(b)} = (a,b)_{L^2}$ when $b \in s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)$, then*

$$\|\ell\| = \|a\|_{s_{t,p'}(\mathscr{H}_1',\mathscr{H}_2')}.$$

The same is true if the $s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)$ spaces are replaced by $s_p^A(\mathscr{H}_1,\mathscr{H}_2)$ spaces.

Proof. We only prove the assertion in the case $t = 1/2$. The general case follows by similar arguments and is left for the reader. Assume that $\ell \in s_p^w(\mathscr{H}_1,\mathscr{H}_2)^*$. Since the map $b \mapsto b^w(x,D)$ is an isometric bijection from $s_p^w(\mathscr{H}_1,\mathscr{H}_2)$ to $\mathscr{I}_p(\mathscr{H}_1,\mathscr{H}_2)$, it follows from Remark 4.7 (1) that for some $S \in \mathscr{I}_{p'}(\mathscr{H}_1,\mathscr{H}_2)$ and each orthonormal basis $(e_j) \in \mathrm{ON}(\mathscr{H}_1)$ that

$$\begin{aligned}&\ell(b) = \mathrm{tr}_{\mathscr{H}_1}(S^* \circ b^w(x,D)) = \sum (b^w(x,D)e_j, Se_j)_{\mathscr{H}_2} \quad \text{and}\\ &\|\ell\| = \|S\|_{\mathscr{I}_{p'}(\mathscr{H}_1,\mathscr{H}_2)},\end{aligned} \tag{4.6}$$

when $b \in s_p^w(\mathscr{H}_1,\mathscr{H}_2)$.

Now let $b \in s_p^w(\mathscr{H}_1,\mathscr{H}_2)$ be an arbitrary finite rank element. Then

$$b = \sum \lambda_j W_{f_j,\varepsilon_j} \quad \text{and} \quad \|b\|_{s_p^w(\mathscr{H}_1,\mathscr{H}_2)} = (2\pi)^{-d/2}\|(\lambda_j)\|_{l^p},$$

for some orthonormal bases $(\varepsilon_j) \in \mathrm{ON}(\mathscr{H}_1')$ and $(f_j) \in \mathrm{ON}(\mathscr{H}_2)$, and some sequence $(\lambda_j) \in l_0^1$. We also let $(e_j) \in \mathrm{ON}(\mathscr{H}_1)$ be the dual basis of (ε_j) and a the Weyl symbol of the operator $T_{\mathscr{H}_2} \circ S \circ T_{\mathscr{H}_1'}$. Then $a \in s_{p'}^w(\mathscr{H}_1,\mathscr{H}_2)$ and $\|a\|_{s_{p'}^w(\mathscr{H}_1,\mathscr{H}_2)} = \|\ell\|$. By straight-forward computations we also get

$$\begin{aligned}\ell(b) &= \mathrm{tr}_{\mathscr{H}_1}(S^* \circ b^w(x,D)) = \sum (b^w(x,D)e_j, Se_j)_{\mathscr{H}_2}\\ &= \sum \lambda_j (f_j, Se_j)_{\mathscr{H}_2} = \sum \lambda_j (f_j, a^w(x,D)\varepsilon_j)_{L^2(\mathbf{R}^d)}\\ &= (2\pi)^{-d/2} \sum \lambda_j (W_{f_j,\varepsilon_j}, a)_{L^2(\mathbf{R}^{2d})} = (2\pi)^{-d/2}(b,a)_{L^2(\mathbf{R}^{2d})}.\end{aligned}$$

Hence $\ell(b) = (2\pi)^{-d/2}(b,a)_{L^2(\mathbf{R}^{2d})}$. The result now follows from these identities and the fact that the set of finite rank elements are dense in $s_p^w(\mathscr{H}_1,\mathscr{H}_2)$. The proof is complete. □

Finally we remark that $\mathscr{S}$ is contained and dense in $s_{t,p}$.

Proposition 4.10. *Assume that $p \in [1,\infty)$, and that $\mathscr{H}_1$ and $\mathscr{H}_2$ are tempered Hilbert spaces on $\mathbf{R}^d$. Then $\mathscr{S}(\mathbf{R}^{2d})$ is dense in $s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)$, $s_p^A(\mathscr{H}_1,\mathscr{H}_2)$, $s_{t,\sharp}(\mathscr{H}_1,\mathscr{H}_2)$ and $s_\sharp^A(\mathscr{H}_1,\mathscr{H}_2)$. Furthermore, $\mathscr{S}(\mathbf{R}^{2d})$ is*

dense in $s_{t,\infty}(\mathscr{H}_1,\mathscr{H}_2)$ and $s^A_\infty(\mathscr{H}_1,\mathscr{H}_2)$ with respect to the weak* topology.

Proof. The result is an immediate consequence of Theorem 4.13 in [48], Remark 4.6 (4) and Corollary 4.3. The proof is complete. □

Remark 4.11. Except for the Hilbert-Schmidt case ($p=2$), it is in general a hard task to find simple characterizations of Schatten-von Neumann classes. Important questions therefore concern of finding embeddings between Schatten-von Neumann classes and well-known function and distribution spaces. Here we recall some of such embeddings:

(i) in Chapter 4 in [35], it is proved that if Q is a unit cube on $\mathbf{R}^d$, $1\le p\le 2$ and f and g are measureable on $\mathbf{R}^d$ and satisfy

$$\Big(\sum_{x_\alpha\in \mathbf{Z}^n} \Big(\int_{x_\alpha+Q} |f(x)|^2\, dx \Big)^{p/2}\Big)^{1/p} < \infty ,$$

and similarily for g, then $f(x)g(D)\in \mathscr{I}_p(L^2)$, or equivalently, $f(x)g(\xi)\in s_{t,p}(\mathbf{R}^{2d})$ when $t=0$;

(ii) Let $B^{p,q}_s(\mathbf{R}^d)$ be the Besov space with parameters $p,q\in [1,\infty]$ and $s\in \mathbf{R}$ (cf. [43,45,47,48] for strict definitions). In [43] sharp embeddings of the form

$$B^{p,q_1}_{s_1}(\mathbf{R}^{2d})\subseteq s^w_p(\mathbf{R}^{2d})\subseteq B^{p,q_2}_{s_2}(\mathbf{R}^{2d})$$

is presented. Here

$$q_1=\min(p,p') \quad \text{and} \quad q_2=\max(p,p'). \tag{4.7}$$

We also remark that the sharp embedding $B^{\infty,1}_s(\mathbf{R}^{2d})\subseteq s_{t,\infty}(\mathbf{R}^{2d})$ for certain choices of t was proved already in [3,4,30,37];

(iii) In [48, Theorem 4.13] it is proved that if $\omega_1,\omega_2\in \mathscr{P}(\mathbf{R}^{2d})$,

$$\omega(x,\xi,\eta,y)=\omega_2(x-ty,\xi+(1-t)\eta)/\omega_1(x+(1-t)y,\xi-t\eta)$$

and $p,q_1,q_2\in [1,\infty]$ satisfy (4.7), then

$$M^{p,q_1}_{(\omega)}(\mathbf{R}^{2d})\subseteq s_{t,p}(\omega_1,\omega_2)\subseteq M^{p,q_2}_{(\omega)}(\mathbf{R}^{2d}). \tag{4.8}$$

In particular, (4.8) covers the Schatten-von Neumann results in [23,36,45], where similar questions are considered in the case $\omega_1=\omega_2=\omega=1$. Furthermore, in [48], embeddings between $s_{t,p}(\omega_1,\omega_2)$ and Besov spaces with $\omega_1=\omega_2$ are established.

5. Young inequalities for weighted Schatten-von Neumann classes

In this section we establish Young type results for dilated convolutions and multiplications on $s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)$ and on $s^A_p(\mathscr{H}_1,\mathscr{H}_2)$, under the assumptions that $\mathscr{H}_1$ and $\mathscr{H}_2$ are appropriate modulation spaces of Hilbert type.

As a preparation for this we prove some technical lemmas, and start with the following classification of Hilbert modulation spaces.

Lemma 5.1. *Assume that $\omega \in \mathscr{P}(\mathbf{R}^{4d})$ is such that $\omega(x,y,\xi,\eta) = \omega(x,\xi)$, $\chi \in \mathscr{S}(\mathbf{R}^d)\setminus 0$ and that $F \in \mathscr{S}'(\mathbf{R}^{2d})$. Then $F \in M^2_{(\omega)}$, if and only if*

$$\|F\| \equiv \Big(\iiint |V_\chi(F(\,\cdot\,,y))(x,\xi)\omega(x,\xi)|^2\,dxdyd\xi\Big)^{1/2}. \tag{5.1}$$

Furthermore, $F \mapsto \|F\|$ in (5.1) defines a norm which is equivalent to any $M^2_{(\omega)}$ norm.

Proof. We may assume that $\|\chi\|_{L^2} = 1$. Let $\chi_1 = \chi\otimes\chi$, and let $\mathscr{F}_1F$ denotes the partial Fourier transform of $F(x,y)$ with respect to the x variable. By Parseval's formula we get

$$\|F\|^2_{M^2_{(\omega)}} = \iiiint |(V_{\chi\otimes\chi}F)(x,y,\xi,\eta)\omega(x,\xi)|^2\,dxdyd\xi d\eta$$

$$\iint \Big(\iint |(\mathscr{F}\big(F\,\chi_1(\,\cdot\, - (x,y))\big)(\xi,\eta)\omega(x,\xi)|^2\,dyd\eta\Big)\,dxd\xi$$

$$\iint \Big(\iint |(\mathscr{F}_1\big(F(\,\cdot\,,z)\,\chi(\,\cdot\, - x)\big)(\xi)\chi(z-y)\omega(x,\xi)|^2\,dydz\Big)\,dxd\xi$$

$$\iint \Big(\int |(\mathscr{F}_1\big(F(\,\cdot\,,z)\,\chi(\,\cdot\, - x)\big)(\xi)\omega(x,\xi)|^2\,dz\Big)\,dxd\xi = \|F\|,$$

where the right-hand side is the same as $\|F\|$ in (5.1). The proof is complete. $\square$

The following lemma is a cornerstone in our further investigations. We omit the proof since the result agrees with [43, Lemma 3.2].

Lemma 5.2. *Assume that $s,t\in\mathbf{R}$ satisfies $(-1)^js^{-2}+(-1)^kt^{-2}=1$, for some choice of $j,k\in\{0,1\}$, and that $a,b\in\mathscr{S}(\mathbf{R}^{2d})$. Also let $T_{j,z}$ for $j\in\{0,1\}$ and $z\in\mathbf{R}^d$ be the operator on $\mathscr{S}(\mathbf{R}^{2d})$, defined by the formula*

$$(T_{0,z}U)(x,y) = (T_{1,z}U)(y,x) = U(x-z,y+z),\quad U\in\mathscr{S}(\mathbf{R}^{2d}).$$

Then

$$\begin{aligned} &A(a(s\,\cdot\,)*b(t\,\cdot\,)) \\ &\qquad = (2\pi)^{d/2}|st|^{-d}\int (T_{j,sz}(Aa))(s^{-1}\,\cdot\,)(T_{k,-tz}(Ab))(t^{-1}\,\cdot\,)\,dz. \end{aligned} \tag{5.2}$$

In Theorem 5.3 concerns dilated convolutions of s^A_p spaces. Here the conditions on the involved weight functions are

$$\begin{aligned} \vartheta(X_1+X_2) &\le C\vartheta_{j_1,1}(t_1X_1)\vartheta_{j_2,2}(t_2X_2) \\ \omega(X_1+X_2) &\le C\omega_{j_1,1}(t_1X_1)\omega_{j_2,2}(t_2X_2) \end{aligned} \tag{5.3}$$

where

$$\omega_{0,k}(X) = \vartheta_{1,k}(-X) = \omega_k(X), \quad \vartheta_{0,k}(X) = \omega_{1,k}(-X) = \vartheta_k(X) \tag{5.4}$$

and

$$(-1)^{j_1} t_1^{-2} + (-1)^{j_2} t_2^{-2} = 1. \tag{5.5}$$

For convenience we also set $u_t = u(\,\cdot\,)$ and $a_{j,t} = a_j(t\,\cdot\,)$.

Theorem 5.3. *Assume that $p_1, p_2, r \in [1,\infty]$ satisfy (0.9), and that $t_1, t_2 \in \mathbf R$ satisfy (5.5), for some choices of $j_1, j_2 \in \{0,1\}$. Also assume that $\omega, \omega_j, \vartheta, \vartheta_j \in \mathscr P(\mathbf R^{2d})$ for $j = 1,2$ satisfy (5.3) and (5.4). Then the mapping $(a_1, a_2) \mapsto a_{1,t_1} * a_{2,t_2}$ on $\mathscr S(\mathbf R^{2d})$, extends uniquely to a continuous mapping from $s^A_{p_1}(1/\omega_1, \vartheta_1) \times s^A_{p_2}(1/\omega_2, \vartheta_2)$ to $s^A_r(1/\omega, \vartheta)$. One has the estimate*

$$\|a_{1,t_1} * a_{2,t_2})\|_{s^A_r(1/\omega,\vartheta)} \le C^d \|a_1\|_{s^A_{p_1}(1/\omega_1,\vartheta_1)} \|a_2\|_{s^A_{p_2}(1/\omega_2,\vartheta_2)}, \tag{5.6}$$

where $C = C_0^2 |t_1|^{-2/p_1} |t_2|^{-2/p_2}$ for some constant C_0 which is independent of t_1, t_2 and d.

Before the proof we note that for the involved spaces in Theorem 5.3 we have

$$s^A_p(1/\omega, \vartheta) \subseteq s^A_p(\mathbf R^{2d}) \subseteq s^A_p(\omega, 1/\vartheta), \quad \text{when} \quad \omega, \vartheta \ge c, \tag{5.7}$$

for some constant $c > 0$. This is an immediate consequence of Remark 4.6 (4) and that the embeddings $M^{2,2}_{(\omega)} \subseteq M^{2,2} = L^2 \subseteq M^{2,2}_{(1/\omega)}$ hold when ω is bounded from below. In particular,

$$s^A_1(1/\omega, \vartheta) \subseteq s^A_1(\mathbf R^{2d}) \subseteq C'_B(\mathbf R^{2d}) \cap \mathscr F C'_B(\mathbf R^{2d}) \cap L^2(\mathbf R^{2d}), \quad \text{when} \quad \omega, \vartheta \ge c, \tag{5.8}$$

where the latter embedding follows from Propositions 1.5 and 1.9 in [44].

Proof. Again we only consider the case $j_1 = 1$ and $j_2 = 0$, i. e. $t^{-2} - s^{-2} = 1$ when $t_1 = s$ and $t_2 = t$. The other cases follows by similar arguments and are left for the reader. We may assume that $W = T^*\mathbf R^d$, and start to prove the theorem in the case $p = q = r = 1$. By Proposition 1.10 and a simple argument of approximations, it follows that we may assume that $a_1 = u$ and $a_2 = v$ are rank one elements in $\mathscr S$ and satisfy

$$\|u\|_{s^A_1(1/\omega_1,\vartheta_1)} \le C, \qquad \|v\|_{s^A_1(1/\omega_2,\vartheta_2)} \le C.$$

for some constant C. If A is the mapping in (0.1), then it follows that $Au = f_1 \otimes \overline f_2$ and $Av = g_1 \otimes \overline g_2$, and

$$\|f_1\|_{M^2_{(\vartheta_1)}} \|f_2\|_{M^2_{(\omega_1)}} \le C_1 \|u\|_{s^A_1(1/\omega_1,\vartheta_1)},$$

$$\|g_1\|_{M^2_{(\vartheta_2)}} \|g_2\|_{M^2_{(\omega_2)}} \le C_1 \|v\|_{s^A_1(1/\omega_2,\vartheta_2)},$$

for some vectors $f_1, f_2, g_1, g_2 \in \mathscr{S}$ such that

$$\|f_1\|_{M^2_{(\vartheta_1)}} \le C_2, \quad \|f_2\|_{M^2_{(\omega_1)}} \le C_2, \quad \|g_1\|_{M^2_{(\vartheta_2)}} \le C_2, \quad \|g_2\|_{M^2_{(\omega_2)}} \le C_2,$$

for some constants C_1 and C_2.

Set

$$F(x,z) = f_2(x/s+sz)g_1(x/t+tz), \qquad G(y,z) = f_1(y/s-sz)g_2(y/t-tz).$$

It follows from (5.2) that

$$A(u_s * v_t)(x,y) = (2\pi)^{d/2}|st|^{-d}\int F(x,z)G(y,z)\,dz.$$

This implies that

$$\begin{aligned}\|u_s * v_t\|_{s_1^A(\omega,\vartheta)} &\le (2\pi)^{d/2}|st|^{-d}\int \|F(\,\cdot\,,z)\|_{M^2_{(\vartheta)}}\|G(\,\cdot\,,z)\|_{M^2_{(\omega)}}\,dz \\ &\le C|st|^{-d}I_1\cdot I_2,\end{aligned} \tag{5.9}$$

where

$$\begin{aligned} I_1 &= \Big(\iiint |V_\chi(F(\,\cdot\,,z))(x,\xi)\vartheta(x,\xi)|^2\,dxdzd\xi\Big)^{1/2} \\ I_2 &= \Big(\iiint |V_\chi(G(\,\cdot\,,z))(x,\xi)\omega(x,\xi)|^2\,dxdzd\xi\Big)^{1/2}. \end{aligned} \tag{5.10}$$

Hence, $I_1 \le C\|F\|_{M^2_{(\vartheta_0)}}$ and $I_2 \le C\|G\|_{M^2_{(\omega_0)}}$ by Lemma 5.1, when $\omega_0(x,y,\xi,\eta) = \omega(x,\xi)$ and $\vartheta_0(x,y,\xi,\eta) = \vartheta(x,\xi)$.

We need to estimate $\|F\|_{M^2_{(\vartheta_0)}}$ and $\|G\|_{M^2_{(\omega_0)}}$. In order to estimate $\|F\|_{M^2_{(\vartheta_0)}}$ we choose the window function $\chi \in \mathscr{S}(\mathbf{R}^{2d})$ as

$$\chi(x,z) = \chi_0(x/s+sz)\chi_0(x/t+tz),$$

for some real-valued $\chi_0 \in \mathscr{S}(\mathbf{R}^d)$. By taking $(x_1/s+sz_1, x_1/t+tz_1)$ as new variables when evaluating $V_\chi F$ we get by formal computations

$$\begin{aligned} &V_\chi F(x,z,\xi,\zeta) \\ &\quad = (2\pi)^{-d}\iint F(x_1,z_1)\chi(x_1-x,z_1-z)e^{-i\langle x_1,\xi\rangle - i\langle z_1,\zeta\rangle}\,dx_1dz_1 \\ &= (2\pi)^{-d}|st|^{-d}\iint \overline{f_2(x_1)}g_1(z_1)\chi_0(x_1-(x/s+sz))\chi_0(z_1-(x/t+tz))\times \\ &\qquad \times e^{-i(\langle t^{-1}z_1 - s^{-1}x_1,\xi\rangle + (st)^{-1}\langle t^{-1}x_1 - s^{-1}z_1,\zeta\rangle}\,dx_1dz_1 \\ &= |st|^{-d}\overline{V_{\chi_0}f_2(s^{-1}x+sz, s^{-1}\xi - (st^2)^{-1}\zeta)}V_{\chi_0}g_1(t^{-1}x+tz, t^{-1}\xi-(s^2t)\zeta). \end{aligned}$$

Furthermore, by (5.3), (5.4) and the fact that $t^{-2} - s^{-2} = 1$, we obtain

$$\vartheta(x,\xi) = \vartheta\big((t^{-2}x+z)-(s^{-2}x+z),(t^{-2}\xi-(st)^{-2}\zeta)-(s^{-2}\xi-(st)^{-2}\zeta)\big)$$
$$\le C\omega_1(s^{-1}x+sz,s^{-1}\xi-(st^2)^{-1}\zeta)\vartheta_2(t^{-1}x+tz,t^{-1}\xi-(s^2t)^{-1}\zeta)$$

A combination of these relations now gives

$$|V_\chi F(x,z,\xi,\zeta)\vartheta(x,\xi)| \le C|st|^{-d}J_1\cdot J_2, \tag{5.11}$$

where

$$J_1 = |V_{\chi_0}f_2(s^{-1}x+sz,s^{-1}\xi-(st^2)^{-1}\zeta)\omega_1(s^{-1}x+sz,s^{-1}\xi-(st^2)^{-1}\zeta)|$$

and

$$J_2 = |V_{\chi_0}g_1(t^{-1}x+tz,t^{-1}\xi-(s^2t)\zeta)\vartheta_2(t^{-1}x+tz,t^{-1}\xi-(s^2t)^{-1}\zeta)|.$$

By applying the L^2 norm and taking

$$s^{-1}x+sz,\quad t^{-1}x+tz,\quad s^{-1}\xi-(st^2)^{-1}\zeta,\quad t^{-1}\xi-(s^2t)^{-1}\zeta$$

as new variables of integration we get

$$\|F\|_{M^2_{(\vartheta)}} \le C|st|^{-2d}\|f_2\|_{M^2_{(\omega_1)}}\|g_1\|_{M^2_{(\vartheta_2)}}. \tag{5.12}$$

By similar computations it also follows that

$$\|G\|_{M^2_{(\omega)}} \le C|st|^{-2d}\|f_1\|_{M^2_{(\vartheta_1)}}\|g_2\|_{M^2_{(\omega_2)}}. \tag{5.13}$$

Hence, a combination of Proposition 4.8, (5.9), (5.10), (5.12) and (5.13) gives

$$\|u_s*v_t\|_{s_1^A(1/\omega,\vartheta)} \le C_1|st|^{-d}\|f_1\|_{M^2_{(\vartheta_1)}}\|f_2\|_{M^2_{(\omega_1)}}\|g_1\|_{M^2_{(\vartheta_2)}}\|g_2\|_{M^2_{(\omega_2)}}$$
$$\le C_2|st|^{-d}\|u\|_{s_1^A(1/\omega_1,\vartheta_1)}\|v\|_{s_1^A(1/\omega_2,\vartheta_2)}.$$

This proves the result in the case $p=q=r=1$.

Next we consider the case $p_1=r=\infty$, which implies that $p_2=1$. Assume that $a\in s_\infty^A(1/\omega_1,\vartheta_1)$ and that $b,c\in\mathscr{S}(\mathbf{R}^{2d})$. Then

$$(a_s*b_t,c) = |s|^{-4d}(a,\widetilde{b}_{t_0}*c_{s_0}),$$

where $\widetilde{b}(X)=\overline{b(-X)}$, $s_0=1/s$ and $t_0=t/s$. We claim that

$$\|\widetilde{b}_{t_0}*c_{s_0}\|_{s_1^A(\omega_1,1/\vartheta_1)} \le C|s^2/t|^{2d}\|b\|_{s_1^A(1/\omega_2,\vartheta_2)}\|c\|_{s_1^A(\omega,1/\vartheta)} \tag{5.14}$$

Admitting this for a while, it follows by duality, using Theorem 4.9 that

$$\|a_s*b_t\|_{s_\infty^A(1/\omega,\vartheta)} \le C|s^2/t|^{2d}s^{-4d}\|a\|_{s_\infty^A(1/\omega_1,\vartheta_1)}\|b\|_{s_1^A(1/\omega_2,\vartheta_2)},$$

which gives (5.6). The result now follows in the case $p_1=r=\infty$ and $p_2=1$ from the fact that $\mathscr{S}$ is dense in $s_1^A(1/\omega_2,\vartheta_2)$. In the same way the result follows in the case $p_2=r=\infty$ and $p_1=1$.

For general $p_1, p_2, r \in [1, \infty]$ the result follows by multi-linear interpolation, using Theorem 4.4.1 in [1] and Remark 4.7 (3).

It remains to prove (5.14) when $b, c \in \mathscr{S}(\mathbf{R}^{2d})$. The condition (5.5) is invariant under the transformation $(t, s) \mapsto (t_0, s_0) = (t/s, 1/s)$. Let

$$\widetilde{\omega} = 1/\omega_1, \quad \widetilde{\vartheta} = 1/\vartheta_1, \qquad \widetilde{\omega}_1 = 1/\omega,$$

$$\widetilde{\vartheta}_1 = 1/\vartheta, \quad \widetilde{\omega}_2 = \vartheta_2 \quad \text{and} \quad \widetilde{\vartheta}_2 = \omega_2.$$

If $X_1 = -(X+Y)/s$ and $X_2 = Y/s$, then it follows that

$$\omega(X_1 + X_2) \le C\vartheta_1(-sX_1)\omega_2(tX_2)$$

and

$$\vartheta(X_1 + X_2) \le C\omega_1(-sX_1)\vartheta_2(tX_2),$$

is equivalent to

$$\widetilde{\omega}(X + Y) \le C\widetilde{\vartheta}_1(-s_0X)\widetilde{\omega}_2(t_0Y)$$

and

$$\widetilde{\vartheta}(X + Y) \le C\widetilde{\omega}_1(-s_0X)\widetilde{\vartheta}_2(t_0Y).$$

Hence, the first part of the proof gives

$$\begin{aligned} \|\widetilde{b}_{t_0} * c_{s_0}\|_{s_1^A(\omega_1, 1/\vartheta_1)} &= \|\widetilde{b}_{t_0} * c_{s_0}\|_{s_1^A(1/\widetilde{\omega}, \widetilde{\vartheta})} \\ &\le C|s_0t_0|^{-2d}\|\widetilde{b}\|_{s_1^A(1/\widetilde{\omega}_2, \widetilde{\vartheta}_2)}\|\widetilde{c}\|_{s_1^A(1/\widetilde{\omega}_1, \widetilde{\vartheta}_1)} \\ &= C|s_0t_0|^{-2d}\|\widetilde{b}\|_{s_1^A(1/\vartheta_2, \omega_2)}\|\widetilde{c}\|_{s_1^A(\omega, 1/\vartheta)} \\ &= C|s_0t_0|^{-2d}\|b\|_{s_1^A(1/\omega_2, \vartheta_2)}\|\widetilde{c}\|_{s_1^A(\omega, 1/\vartheta)}, \end{aligned}$$

and (5.14) follows. □

Remark 5.4. A proof without any use of interpolation in the case of trivial weight is presented in Section 2.3 in [41].

There is a natural generalization of Theorem 5.3 to the case of more than two factors in the convolution. We recall that the corresponding Young condition (0.9) for the exponents when we have convolutions with N functions is

$$(0.9)' \quad {p_1}^{-1} + \dots + {p_N}^{-1} = N - 1 + r^{-1}, \qquad 1 \le p_1, \dots, p_N, r \le \infty.$$

The condition on the involved weight functions is

$$(5.3)' \quad \begin{aligned} \vartheta(X_1 + \dots + X_N) &\le C\vartheta_{j_1,1}(t_1X_1)\cdots\vartheta_{j_N,N}(t_NX_N) \\ \omega(X_1 + \dots + X_N) &\le C\omega_{j_1,1}(t_1X_1)\cdots\omega_{j_N,N}(t_NX_N) \end{aligned}$$

where

$$(5.4)' \quad \omega_{0,k}(X)=\vartheta_{1,k}(-X)=\omega_k(X),\quad \vartheta_{0,k}(X)=\omega_{1,k}(-X)=\vartheta_k(X)$$

and

$$(5.5)' \qquad (-1)^{j_1}t_1^{-2}+\cdots+(-1)^{j_N}t_N^{-2}=1.$$

Theorem 5.3′. *Assume that $p_1,\dots,p_N,r\in[1,\infty]$ satisfy $(0.9)'$, and that $t_1,\dots,t_N\in\mathbf{R}$ satisfy $(5.5)'$, for some choices of $j_1,\dots,j_N\in\{0,1\}$. Also assume that $\omega,\omega_j,\vartheta,\vartheta_j\in\mathscr{P}(\mathbf{R}^{2d})$ for $j=1,\dots,N$ satisfy $(5.3)'$ and $(5.4)'$. Then the mapping $(a_1,\dots,a_N)\mapsto a_{1,t_1}*\cdots*a_{N,t_N}$ on $\mathscr{S}(\mathbf{R}^{2d})$ extends uniquely to a continuous mapping from $s^A_{p_1}(1/\omega_1,\vartheta_1)\times\cdots\times s^A_{p_N}(1/\omega_N,\vartheta_N)$ to $s^A_r(1/\omega,\vartheta)$. One has the estimate*

$$(5.6)' \qquad \begin{aligned}&\|a_{1,t_1}*\cdots*a_{N,t_N}\|_{s^A_r(1/\omega,\vartheta)}\\ &\qquad\le C^d\|a_1\|_{s^A_{p_1}(1/\omega_1,\vartheta_1)}\cdots\|a_N\|_{s^A_{p_N}(1/\omega_N,\vartheta_N)},\end{aligned}$$

where $C=C_0^N|t_1|^{-2/p_1}\cdots|t_N|^{-2/p_N}$ for some constant C_0 which is independent of N, $t_1,\dots,t_N$ and d.

For the proof we need the following lemma.

Lemma 5.5. *Assume that $\rho,t_1,\dots,t_N\in\mathbf{R}\setminus 0$ fulfills $(5.5)'$ and $\rho^{-2}+(-1)^{j_N}t_N^{-2}=1$, and set $t_j'=t_j/\rho$,*

$$\widetilde{\omega}(X)=\inf\omega_{j_1,1}(t_1'X_1)\cdots\omega_{j_{N-1},N-1}(t_{N-1}'X_{N-1})$$

and

$$\widetilde{\vartheta}(X)=\inf\vartheta_{j_1,1}(t_1'X_1)\cdots\vartheta_{j_{N-1},N-1}(t_{N-1}'X_{N-1}),$$

where the infima are taken over all $X_1,\dots X_{N-1}$ such that $X=X_1+\cdots X_{N-1}$. Then the following is true:

(1) *$\widetilde{\omega},\widetilde{\vartheta}\in\mathscr{P}(\mathbf{R}^{2d})$;*

(2) *for each $X_1,\dots X_{N-1}\in\mathbf{R}^{2d}$ it holds*

$$\widetilde{\omega}(X_1+\cdots+X_{N-1})=\omega_{j_1,1}(t_1'X_1)\cdots\omega_{j_{N-1},N-1}(t_{N-1}'X_{N-1})$$

and

$$\widetilde{\vartheta}(X_1+\cdots+X_{N-1})=\vartheta_{j_1,1}(t_1'X_1)\cdots\vartheta_{j_{N-1},N-1}(t_{N-1}'X_{N-1});$$

(3) *if C is the same as in $(5.3)'$, then for each $X,Y\in\mathbf{R}^{2d}$ it holds*

$$\omega(X+Y)\le C\widetilde{\omega}(\rho X)\omega_N(t_NY),\quad\text{and}\quad \vartheta(X+Y)\le C\widetilde{\vartheta}(\rho X)\vartheta_N(t_NY).$$

Proof. The assertion (2) follows immediately from the definitions of $\widetilde{\omega}$ and $\widetilde{\vartheta}$, and (3) is an immediate consequence of $(5.3)'$.

In order to prove (3) we assume that $X = X_1 + \cdots + X_{N-1}$. Since $\omega_{j_1,1} \in \mathscr{P}(\mathbf{R}^{2d})$, it follows that

$$\widetilde{\omega}(X+Y) \le \omega_{j_1,1}(t_1'(X_1+Y)) \cdots \omega_{j_{N-1},N-1}(t_{N-1}'X_{N-1}) \\ \le \omega_{j_1,1}(t_1'X_1) \cdots \omega_{j_{N-1},N-1}(t_{N-1}'X_{N-1})v(Y),$$

for some $v \in \mathscr{P}(\mathbf{R}^{2d})$. By taking the infimum over all representations $X = X_1 + \cdots + X_N$, the latter inequality becomes $\widetilde{\omega}(X+Y) \le \widetilde{\omega}(X)v(Y)$. This implies that $\widetilde{\omega} \in \mathscr{P}(\mathbf{R}^{2d})$, and in the same way it follows that $\widetilde{\vartheta} \in \mathscr{P}(\mathbf{R}^{2d})$. The proof is complete. $\square$

Proof of Theorem 5.3′. We may assume that $N > 2$ and that the theorem is already proved for lower values on N. The condition on t_j is that $c_1t_1^{-2} + \cdots + c_Nt_N^{-2} = 1$, where $c_j \in \{\pm 1\}$. For symmetry reasons we may assume that $c_1t_1^{-2} + \cdots + c_{N-1}t_{N-1}^{-2} = \rho^{-2}$, where $\rho > 0$. Let $t_j' = t_j/\rho$, $\widetilde{\omega}$ and $\widetilde{\vartheta}$ be the same as in Lemma 5.5, and let $r_1 \in [1,\infty]$ be such that $1/r_1 + 1/p_N = 1 + 1/r$. Then $c_1(t_1')^{-2} + \cdots + c_{N-1}(t_{N-1}')^{-2} = 1$, $r_1 \ge 1$ since $p_N \le r$, and

$$1/p_1 + \cdots + 1/p_{N-1} = N - 2 + 1/r_1.$$

By the induction hypothesis and Lemma 5.5 (2) it follows that

$$b = a_{1,t_1'} * \cdots * a_{N-1,t_{N-1}'} = \rho^{d(2N-4)}(a_{1,t_1} * \cdots * a_{N-1,t_{N-1}})(\cdot/\rho)$$

makes sense as an element in $s_{r_1}^A(1/\widetilde{\omega},\widetilde{\vartheta})$, and

$$\|b\|_{s_{r_1}^A(1/\widetilde{\omega},\widetilde{\vartheta})} \le C \prod_{j=1}^{N-1} |t_j'|^{-2d/p_j} \|a\|_{s_{p_j}^A(1/\omega_j,\vartheta_j)},$$

for some constant C. Since $1/r_1 + 1/p_N = 1 + 1/r$, it follows from Lemma 5.5 (3) that $b_\rho * a_{N,t_N}$ makes sense as an element in $s_r^A(1/\omega,\vartheta)$, and

$$\|(a_{1,t_1} * \cdots * a_{N-1,t_{N-1}}) * a_{N,t_N}\|_{s_r^A(1/\omega,\vartheta)} = \rho^{-d(2N-4)}\|b_\rho * a_{N,t_N}\|_{s_r^A(1/\omega,\vartheta)} \\ \le C_1 \|a_1\|_{s_{p_1}^A(1/\omega_1,\vartheta_1)} \cdots \|a_N\|_{s_{p_N}^A(1/\omega_N,\vartheta_N)},$$

where

$$C_1 = C\rho^{d(4-2N-2/r_1)}|t_N|^{-2d/p_N} \prod_{j=1}^{N-1} |t_j'|^{-2d/p_j} = C \prod_{j=1}^{N} |t_j|^{-2d/p_j}.$$

This proves the extension assertions. The uniqueness as well as the symmetri assertions follow from the facts that $\mathscr{S}$ is dense in s_p^A when $p < \infty$ and dense in s_∞^A with respect to the weak* topology, and that at most one p_j is equal to infinity due to the Young condition. The proof is complete. $\square$

The first part of the following result follows by combining Theorem 5.3′, Proposition 4.5 and

$$\mathscr{F}_\sigma(a_1 * \cdots * a_N) = \pi^{dN}(\mathscr{F}_\sigma a_1)\cdots(\mathscr{F}_\sigma a_N), \tag{5.15}$$

when $a_1,\dots,a_N \in \mathscr{S}(\mathbf{R}^{2d})$. Here the condition $(5.5)'$ is replaced by

$$(-1)^{j_1}t_1^2 + \cdots + (-1)^{j_N}t_N^2 = 1. \tag{5.16}$$

Theorem 5.6. *Assume that $p_1,\dots,p_N,r \in [1,\infty]$ satisfy $(0.9)'$, and that $t_1,\dots,t_N \in \mathbf{R}$ satisfy (5.16), for some choices of $j_1,\dots,j_N \in \{0,1\}$. Also assume that $\omega,\omega_j,\vartheta,\vartheta_j \in \mathscr{P}(\mathbf{R}^{2d})$ for $j=1,\dots,N$ satisfy $(5.3)'$ and $(5.4)'$. Then the mapping $(a_1,\dots,a_N) \mapsto a_{1,t_1}\cdots a_{N,t_N}$ on $\mathscr{S}(\mathbf{R}^{2d})$, where $a_{j,t_j}(X) = a_j(t_jX)$, $1 \le j \le N$, extends uniquely to a continuous mapping from $s^A_{p_1}(1/\omega_1,\vartheta_1)\times\cdots\times s^A_{p_N}(1/\omega_N,\vartheta_N)$ to $s^A_r(1/\omega,\vartheta)$. One has the estimate*

$$\begin{aligned} &\|a_{1,t_1}\cdots a_{N,t_N}\|_{s^A_r(1/\omega,\vartheta)} \\ &\qquad \le C^d\|a_1\|_{s^A_{p_1}(1/\omega_1,\vartheta_1)}\cdots\|a_N\|_{s^A_{p_N}(1/\omega_N,\vartheta_N)}. \end{aligned} \tag{5.17}$$

where $C = C_0^N|t_1|^{-2/p_1'}\cdots|t_N|^{-2/p_N'}$ for some constant C_0 which is independent of N, $t_1,\dots,t_N$ and d.

Moreover, the product is positive semi-definite in the sense of Definition 1.5, if this is true for each factor.

Proof. When verifying the positivity statement we may argue by induction as in the proof of Theorem 5.3′. This together with Proposition 1.6 and some simple arguments of approximation shows that it suffices to prove that a_sb_t is positive semi-definite when $\pm s^2 \pm t^2 = 1$, $st \neq 0$, and $a,b \in \mathscr{S}(\mathbf{R}^{2d})$ are σ-positive rank-one element.

We write

$$a_sb_t = \pi^{-d}\mathscr{F}_\sigma(\mathscr{F}_\sigma a_s * \mathscr{F}_\sigma b_t) = \pi^{-d}|st|^{-2d}\mathscr{F}_\sigma((\mathscr{F}_\sigma a)_{1/s} * (\mathscr{F}_\sigma b)_{1/t}).$$

If we set for any $U \in \mathscr{S}(V\oplus V)$,

$$U_{0,z}(x,y) = U_{1,z}(-y,-x) = U(x+z,y+z),$$

then it follows from Lemmas 1.4 and 5.2 that

$$A(a_sb_t)(x,y) = (2/\pi)^{d/2}|st|^{-d}\int (Aa)_{j,z/s}(sx,sy)(Ab)_{k,-z/t}(tx,ty)\,dz,$$

for some choice of $j,k \in \{0,1\}$. Since $a,b \in C_+$ are rank-one elements, it follows that the integrand is of the form $\phi_z(x)\otimes\overline{\phi_z(y)}$ in all these cases. This proves that $A(a_sb_t)$ is a positive semi-definite operator. $\square$

The following two theorems follow immediately from Proposition 4.5, Theorem 5.3′ and Theorem 5.6. Here the condition $(5.4)'$ is replaced by

$$\omega_{0,k}(X) = \vartheta_{1,k}(X) = \omega_k(X), \quad \vartheta_{0,k}(X) = \omega_{1,k}(X) = \vartheta_k(X). \tag{5.4$''$}$$

Theorem 5.7. *Assume that* $p_1,\dots,p_N,r\in[1,\infty]$ *satisfy* (0.9)′*, and that* $t_1,\dots,t_N\in\mathbf R$ *satisfy* (5.5)′*, for some choices of* $j_1,\dots,j_N\in\{0,1\}$*. Also assume that* $\omega,\omega_j,\vartheta,\vartheta_j\in\mathscr P(\mathbf R^{2d})$ *for* $j=1,\dots,N$ *satisfy* (5.3)′ *and* (5.4)″*. Then the mapping* $(a_1,\dots,a_N)\mapsto a_{1,t_1}*\cdots*a_{N,t_N}$ *on* $\mathscr S(\mathbf R^{2d})$*, where* $a_{j,t_j}(X)=a_j(t_jX)$, $1\le j\le N$*, extends uniquely to a continuous mapping from*

$$s^w_{p_1}(1/\omega_1,\vartheta_1)\times\cdots\times s^w_{p_N}(1/\omega_N,\vartheta_N)$$

to $s^w_r(1/\omega,\vartheta)$*. One has the estimate*

$$\begin{aligned}\|a_{1,t_1}*\cdots*a_{N,t_N}\|_{s^w_r(1/\omega,\vartheta)}\\ \le C^d\|a_1\|_{s^w_{p_1}(1/\omega_1,\vartheta_1)}\cdots\|a_N\|_{s^w_{p_N}(1/\omega_N,\vartheta_N)},\end{aligned}\tag{5.18}$$

where $C=C_0^N|t_1|^{-2/p_1}\cdots|t_N|^{-2/p_N}$ *for some constant* C_0 *which is independent of* N, $t_1,\dots,t_N$ *and* d.

Moreover, if $a_j^w(x,D)\ge0$ *for each* $1\le j\le N$*, then* $(a_{1,t_1}*\cdots*a_{N,t_N})^w(x,D)\ge0$.

Theorem 5.8. *Assume that* $p_1,\dots,p_N,r\in[1,\infty]$ *satisfy* (0.9)′*, and that* $t_1,\dots,t_N\in\mathbf R$ *satisfy* (5.16)*, for some choices of* $j_1,\dots,j_N\in\{0,1\}$*. Also assume that* $\omega,\omega_j,\vartheta,\vartheta_j\in\mathscr P(\mathbf R^{2d})$ *for* $j=1,\dots,N$ *satisfy* (5.3)′ *and* (5.4)″*. Then the mapping* $(a_1,\dots,a_N)\mapsto a_{1,t_1}\cdots a_{N,t_N}$ *on* $\mathscr S(\mathbf R^{2d})$*, where* $a_{j,t_j}(X)=a_j(t_jX)$, $1\le j\le N$*, extends uniquely to a continuous mapping from*

$$s^w_{p_1}(1/\omega_1,\vartheta_1)\times\cdots\times s^w_{p_N}(1/\omega_N,\vartheta_N)$$

to $s^w_r(1/\omega,\vartheta)$*. One has the estimate*

$$\begin{aligned}\|a_{1,t_1}\cdots a_{N,t_N}\|_{s^w_r(1/\omega,\vartheta)}\\ \le C^d\|a_1\|_{s^w_{p_1}(1/\omega_1,\vartheta_1)}\cdots\|a_N\|_{s^w_{p_N}(1/\omega_N,\vartheta_N)},\end{aligned}\tag{5.19}$$

where $C=C_0^N|t_1|^{-2/p_1'}\cdots|t_N|^{-2/p_N'}$ *for some constant* C_0 *which is independent of* N, $t_1,\dots,t_N$ *and* d.

Remark 5.9. Theorem 5.7 can also be generalized to involve $s_{t,p}$ spaces, for general $t\in\mathbf R$.

In fact, assume that p_j, r, t_j, ω, ω_j, ϑ and ϑ_j for $1\le j\le N$ are the same as in Theorems 5.7 and 5.8. Also assume that $t\in\mathbf R$, and let $\tau_k=t$ when $j_k=0$ and $\tau_k=1-t$ when $j_k=1$. (The numbers j_k are the same as in (5.5)′.)

Then the mapping $(a_1,\dots,a_N)\mapsto a_{1,t_1}*\cdots a_{N,t_N}$ on $\mathscr S(\mathbf R^{2d})$, extends uniquely to a continuous mapping from

$$s_{\tau_1,p_1}(1/\omega_1,\vartheta_1)\times\cdots\times s_{\tau_N,p_N}(1/\omega_N,\vartheta_N)$$

to $s_{t,r}(1/\omega,\vartheta)$. Furthermore it holds

$$\begin{aligned}\|a_{1,t_1}*\cdots*a_{N,t_N}\|_{s^A_r(1/\omega,\vartheta)}\\ \le C^d\|a_1\|_{s_{\tau_1,p_1}(1/\omega_1,\vartheta_1)}\cdots\|a_N\|_{s_{\tau_N,p_N}(1/\omega_N,\vartheta_N)}.\end{aligned}\tag{5.20}$$

where $C = C_0^N|t_1|^{-2a/p_1}\cdots|t_N|^{-2/p_N}$ for some constant C_0 which is independent of N, $t_1,\dots,t_N$ and d.

Moreover, if $(a_j)_{\tau_j}(x,D)\ge 0$ for each $1\le j\le N$, then $(a_{1,t_1}*\cdots*a_{N,t_N})_t(x,D)\ge 0$.

When proving this we first assume that $a_1,\dots,a_N\in\mathscr S$. By Proposition 4.5 we get

$$
\begin{aligned}
\|a_{1,t_1}*\cdots*a_{N,t_N}\|_{s_{t,r}(1/\omega,\vartheta)} &= \|e^{-i(t-1/2)\langle D_x,D_\xi\rangle}(a_{1,t_1}*\cdots*a_{N,t_N})\|_{s_r^w(1/\omega,\vartheta)}\\
&= \|b_1*\cdots*b_N\|_{s_r^w(1/\omega,\vartheta)},
\end{aligned}
$$

where

$$
b_k = e^{-i(-1)^{j_k}(t-1/2)\langle D_x,D_\xi\rangle/t_k^2}(a_k(t_k\,\cdot\,)) = (e^{-i(-1)^{j_k}(t-1/2)\langle D_x,D_\xi\rangle}a_k)(t_k\,\cdot\,).
$$

Hence by Theorem 5.7 we get

$$
\|a_{1,t_1}*\cdots*a_{N,t_N}\|_{s_{t,r}(1/\omega,\vartheta)}\le CI_1\cdots I_N,
$$

where

$$
I_k = \|e^{-i(-1)^{j_k}(t-1/2)\langle D_x,D_\xi\rangle}a_k\|_{s_{p_k}^w(1/\omega_k,\vartheta_k)} = \|a_k\|_{s_{\tau_k,p_k}(1/\omega_k,\vartheta_k)}.
$$

This gives (5.20).

The result now follows from (5.20) and the fact that $\mathscr S$ is dense in $s_{t,p}(\omega_1,\omega_2)$ when $p<\infty$, and dense in $s_{t,\infty}(\omega_1,\omega_2)$ with respect to the weak* topology.

Next we consider elements in $s_1^A(1/v,v)$, where $v=\check v\in\mathscr P(\mathbf R^{2d})$ is submultiplicative. We note that each element in $s_1^A(1/v,v)$ is a continuous function which turns to zero at infinity, since (5.8) shows that $s_1^A(1/v,v)\subseteq C_B(\mathbf R^{2d})$.

It follows that any product of odd numbers of elements in $s_1^A(1/v,v)$ are again in $s_1^A(1/v,v)$. In fact, assume that $a_1,\dots,a_N\in s_1^A(1/v,v)$, $|\alpha|$ is odd, and that $t_j=1$. Then it follows from Theorem 5.6 that $a_1^{\alpha_1}\cdots a_N^{\alpha_N}\in s_1^A(1/v,v)$, and

$$
\|a_1^{\alpha_1}\cdots a_N^{\alpha_N}\|_{s_1^A(1/v,v)}\le C_0^{d|\alpha|}\prod\|a_j\|^{\alpha_j}_{s_1^A(1/v,v)}, \tag{5.21}
$$

for some constant C_0 which is independent of α and d.

Furthermore, if in addition $a_1,\dots,a_N$ are σ-positive in the sense of Definition 1.5, then the same is true for $a_1^{\alpha_1}\cdots a_N^{\alpha_N}$. The following result is an immediate consequence of these observations.

Proposition 5.10. *Assume that $a_1,\dots,a_N\in s_1^A(1/v,v)$, where $v=\check v\in\mathscr P(\mathbf R^{2d})$ is submultiplicative, C_0 is the same as in (5.21), and assume that $R_1,\dots,R_N>0$. Also assume that f,g are odd analytic functions from the polydisc*

$$
\{\, z\in\mathbf C^N\,;\,|z_j|<C_0R_j\,\}
$$

to $\mathbf{C}$*, with expansions*

$$f(z)=\sum_\alpha c_\alpha z^\alpha \quad \text{and} \quad g(z)=\sum_\alpha |c_\alpha| z^\alpha .$$

Then $f(a)=f(a_1,\dots,a_N)$ *is well-defined and belongs to* $s_1^A(1/v,v)$*. One has the estimate*

$$\|f(a)\|_{s_1^A(1/v,v)} \le g(C_0\|a_1\|_{s_1^A(1/v,v)},\dots,C_0\|a_N\|_{s_1^A(1/v,v)}).$$

If in addition $a_1,\dots,a_N\in C_+(\mathbf{R}^{2d})$*, then* $g(a)\in C_+(\mathbf{R}^{2d})$*.*

An open question for the author is whether the Theorems 5.3–5.8 and Remark 5.9 are true for other dilations. This might then lead to improvements of Proposition 5.10. In this context we note that $s_1^A(\mathbf{R}^{2d})$, and therefore $s_\infty^A(\mathbf{R}^{2d})$ by duality, are not stable under dilations (see Proposition 2.1.12 in [41] or Proposition 5.4 in [44]). We refer to [43,44] for a further properties of σ-positive functions and distributions.

For rank one elements we also have the following positivity result.

Proposition 5.11. *Assume that* $v,v_1\in\mathscr P(\mathbf{R}^{2d})$ *are submultiplicative and fulfill* $v_1=v(\,\cdot\,/\sqrt2\,)$*,* $u\in s_\infty^w(1/\omega,\omega)$ *is an element of rank one, and let* $a(X)=|u(X/\sqrt2)|^2$*. Then* $a\in s_1^w(1/v_1,v_1)$*, and* $a^w(x,D)\ge 0$*.*

Proof. Since u is rank one, it follows from Proposition 4.5 that $u,\overline{u}\in s_1^w(1/v,v)$, which implies that $a\in s_1^w(1/v_1,v_1)$ in view of Theorem 5.8. The result now follows from this fact and Proposition 4.10 in [43]. $\square$

We finish the section by applying our results on Toeplitz operators. The following result is can be considered as a parallel result to the recent results in [50], especially to Theorem 3.1 and Theorem 3.5 in [50]. It also generalizes Proposition 4.5 in [45].

Theorem 5.12. *Assume that* $p\in[1,\infty]$ *and* $\omega,\omega_0,\vartheta,\vartheta_j\in\mathscr P(\mathbf{R}^{2d})$ *for* $j=0,1,2$ *satisfy*

$$\omega(X_1-X_2)\le C\omega_0(\sqrt2\,X_1)\vartheta_2(X_2)$$

and

$$\vartheta(X_1-X_2)\le C\vartheta_0(\sqrt2\,X_1)\vartheta_1(X_2)$$

Then the definition of $\operatorname{Tp}_{h_1,h_2}(a)$ *extends uniquely to each* $a\in\mathscr S'(\mathbf{R}^{2d})$ *and* $h_j\in M^2_{(\vartheta_j)}$ *for* $j=1,2$ *such that* $a(\sqrt2\,\cdot\,)\in s_p^w(1/\omega_0,\vartheta_0)$*, and for some constant* C *it holds*

$$\|\operatorname{Tp}_{h_1,h_2}(a)\|_{\mathscr I_p(M^2_{(1/\omega)},M^2_{(\vartheta)})}\le C\|a(\sqrt2\,\cdot\,)\|_{s_p^w(1/\omega_0,\vartheta_0)}\|h_1\|_{M^2_{(\vartheta_1)}}\|h_2\|_{M^2_{(\vartheta_2)}}.$$

Furthermore, if $h_1=h_2$ *and* $b^w(x,D)\ge0$*, where* $b=a(\sqrt2\,\cdot\,)$*, then* $\operatorname{Tp}_{h_1,h_2}(a)\ge 0$*.*

Proof. Since $W_{h_2,h_1}\in s_1^w(1/\vartheta_1,\vartheta_2)$, the result is an immediate consequence of (1.15) and Theorem 5.7. $\square$

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